SCATTERING THEORY FOR THE DEFOCUSING FOURTH-ORDER SCHRÖDINGER EQUATION

CHANGXING MIAO AND JIQIANG ZHENG

ABSTRACT. In this paper, we study the global well-posedness and scattering theory for the defocusing fourth-order nonlinear Schrödinger equation (FNLS) $iu_t + \Delta^2 u + |u|^p u = 0$ in dimension $d \geqslant 9$. We prove that if the solution u is apriorily bounded in the critical Sobolev space, that is, $u \in L_t^\infty(I; \dot H_x^{s_c}(\mathbb{R}^d))$ with all $s_c := \frac{d}{2} - \frac{4}{p} \geqslant 1$ if p is an even integer or $s_c \in [1, 2+p)$ otherwise, then u is global and scatters. The impetus to consider this problem stems from a series of recent works for the energy-supercritical and energy-subcritical nonlinear Schrödinger equation (NLS) and nonlinear wave equation (NLW). We will give a uniform way to treat the energy-subcritical, energy-critical and energy-supercritical FNLS, where we utilize the strategy derived from concentration compactness ideas to show that the proof of the global well-posedness and scattering is reduced to exclude the existence of three scenarios: finite time blowup; soliton-like solution and low to high frequency cascade. Making use of the No-waste Duhamel formula, we deduce that the energy or mass of the finite time blow-up solution is zero and so get a contradiction. Finally, we adopt the double Duhamel trick, the interaction Morawetz estimate and interpolation to kill the last two scenarios.

Key Words: Fourth-order Schrödinger equation; scattering theory; Strichartz estimate; critical regularity; concentration compactness.
AMS Classification: 35P25, 35Q55, 47J35.

1. Introduction

This paper is mainly concerned with the Cauchy problem of the defocusing fourth-order Schrödinger equation (FNLS)

$$\begin{cases} iu_t + \Delta^2 u + f(u) = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ d \geqslant 9, \\ u(0,x) = u_0(x) \in \dot H^{s_c}(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where $f(u) = |u|^p u,\ u$ is a complex-valued function defined in $\mathbb{R}^{1+d}$, Δ is the Laplacian in $\mathbb{R}^d$, and $s_c := \frac{d}{2} - \frac{4}{p}$.

If the solution u of (1.1) has sufficient decay at infinity and smoothness, it conserves mass

$$M(u) = \int_{\mathbb{R}^d} |u(t,x)|^2 dx = M(u_0) \tag{1.2}$$

and energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\Delta u|^2 dx + \frac{1}{p+2} \int_{\mathbb{R}^d} |u(t,x)|^{p+2} dx = E(u_0). \tag{1.3}$$

As similarly explained in [10], the above quantities are also conserved for the energy solutions $u \in C_t^0(\mathbb{R}, H^2(\mathbb{R}^d))$. We call $\dot H_x^2(\mathbb{R}^d)$ the energy space.

The equation (1.1) has the scaling invariance symmetry:

$$u(t,x) \mapsto \lambda^{\frac{4}{p}}\ u(\lambda^4\ t, \lambda\ x),\ \forall\ \lambda > 0 \tag{1.4}$$

in the sense that both the equation and the $\dot{H}^{s_c}$-norm are invariant under the scaling transformation:

$$\|u_\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u\|_{\dot{H}^{s_c}(\mathbb{R}^d)}.$$

We call FNLS (1.1) the energy-subcritical when $p < \frac{8}{d-4}$, which corresponds to $s_c < 2$, in particular, it is called the mass-critical when $p = \frac{8}{d}$, corresponding to $s_c = 0$; (1.1) refers to energy-critical when $d \geqslant 5$ and $p = \frac{8}{d-4}$, corresponding to $s_c = 2$; and (1.1) refers to energy-supercritical when $p > \frac{8}{d-4}$, corresponding to $s_c > 2$.

Fourth-order Schrödinger equations have been introduced by Karpman [12] and Karpman and Shagalov [13] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations are written as

$$i\partial_t u + \Delta^2 u + \varepsilon\Delta u + f(|u|^2)u = 0, \tag{1.5}$$

where $\varepsilon \in \{\pm 1, 0\}$. Such equations have been studied from the mathematical viewpoint in Fibich, Ilan and Papanicolaou [9] who describe various properties of the equaion in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Related reference is [1] by Ben-Artzi, Koch, and Saut, which gives sharp dispersive estimates for the biharmonic Schrödinger operator which lead to the Strichartz estimates for the fourth-order Schrödinger equation, see also [27, 31, 32]. Guo and Wang [11] who prove global well-posedness and scattering in H^s for small data. For other special fourth order nonlinear Schrödinger equation, please refer to [35, 42, 43]. For FNLS (1.1), the defocusing energy-critical case with nonlinearity given by $f(u) = |u|^{\frac{8}{d-4}}u$ was handled by Pausader [31, 32] in dimension $d = 8$, in which case the nonlinearity is cubic, and Miao, Xu and Zhao [26] in dimension $d \geqslant 9$. We also refer to Miao, Xu and Zhao [25] and Pausader [33] for the focusing case with radially symmetrical initial data. For the defocusing mass-critical case with nonlinearity given by $f(u) = |u|^{\frac{8}{d}}u$, we refer to Pausader and Shao [34], Xia and Pausader [41].

On the other hand, the global well-posedness and scattering theory for the nonlinear Schrödinger equations (NLS)

$$i\partial_t u - \Delta u \pm |u|^p u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \tag{1.6}$$

have been intensively studied recently, most notably by Bourgain [2], Colliander, Keel, Staffilanni, Takaoka and Tao [4], Kenig and Merle [15] and Killip and Visan [20] and Visan [39, 40] for the energy-critical case and Tao, Visan and Zhang [37], Killip, Tao and Visan [18], Killip, Visan and Zhang [24] and Dodson [5–8] for the mass-critical case.

So far, there is no technology for treating large-data NLS without some a priori control of a critical norm other than the energy-critical NLS and mass-critical NLS. In [16], Kenig-Merle first showed that if the radial solution u to NLS obeys $u \in L_t^\infty(I; \dot{H}^{s_c}(\mathbb{R}^3))$ with $s_c = \frac{1}{2}$, then u is global and scatters, where they utilized their concentration compactness technique as in [15], together with the Lin-Strauss Morawetz inequality which scales like $\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^d)$ and is scaling-critical in this case. Thereafter, Killip–Visan [19]

proved such result for NLS in some energy-supercritical regime. In particular, they deal with the case of a cubic nonlinearity for $d \geqslant 5$, along with some other cases for which $s_c > 1$ and $d \geqslant 5$, where the restriction to high dimensions comes from the double Duhamel trick. Recently, Murphy [30] considers the energy-subcritical NLS by making use of the tool "long time Strichartz estimate" developed by Dodson [5] for almost periodic solutions in the mass-critical setting.

In this paper, we will give a uniform way to treat the energy-subcritical, energy-critical and energy-supercritical FNLS in dimension $d \geqslant 9$. We remark that the arguments in this paper also work for the energy-critical and some energy-subcritical NLS in dimension $d \geqslant 5$.

Now we introduce some background materials.

Definition 1.1 (solution). *A function $u: I \times \mathbb{R}^d \to \mathbb{C}$ on a nonempty time interval $I \subset \mathbb{R}$ is a strong solution to (1.1) if $u \in C_t(K; \dot{H}^{s_c}_x(\mathbb{R}^d)) \cap L^{\frac{d+4}{4}p}_{t,x}(K \times \mathbb{R}^d)$ for any compact interval $K \subset I$ and for any $t, t_0 \in I$, it obeys the Duhamel formula:*

$$u(t,x) = e^{i(t-t_0)\Delta^2}u(t_0) - i\int_{t_0}^{t} e^{i(t-s)\Delta^2} f(u(s))ds. \tag{1.7}$$

We say that the interval I is the lifespan of u. We call u a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. In particular, if $I = \mathbb{R}$, then we say that u is a global solution.

The solution lies in the space $L^{\frac{(d+4)p}{4}}_{t,x}(I \times \mathbb{R}^d)$ locally in time is natural since by Strichartz estimate (see Proposition 2.1 below), the linear flow always lies in this space. Also, if a solution u to (1.1) is global, with $\|u\|_{L^{\frac{d+4}{4}p}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} < +\infty$, then it scatters in both time directions in the sense that there exist unique $v_\pm \in \dot{H}^{s_c}_x(\mathbb{R}^d)$ such that

$$\left\|u(t) - e^{it\Delta^2}v_\pm\right\|_{\dot{H}^{s_c}_x(\mathbb{R}^d)} \longrightarrow 0, \quad \text{as} \quad t \longrightarrow \pm\infty. \tag{1.8}$$

In view of this, we define

$$S_I(u) = \|u\|_{L^{\frac{d+4}{4}p}_{t,x}(I\times\mathbb{R}^d)} \tag{1.9}$$

as the scattering size of u.

Closely associated with the notion of scattering is the notion of blow-up:

Definition 1.2 (Blow-up). *Let $u: I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to (1.1). If there exists a time $t_0 \in I$ such that $S_{[t_0, \sup I)}(u) = +\infty$, then we say that the solution u blows up forward in time. Similarly, if there exists a time $t_0 \in I$ such that $S_{(inf\ I, t_0]}(u) = +\infty$, then we say that $u(t,x)$ blows up backward in time.*

Now we state our main result.

Theorem 1.1. *Assume that $d \geqslant 9$, and $s_c \geqslant 1$ if p is an even integer or $s_c \in [1, 2+p)$ otherwise. Let $u: I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to (1.1) such that*

$$\|u\|_{L^\infty_t(I; \dot{H}^{s_c}_x(\mathbb{R}^d))} < +\infty. \tag{1.10}$$

Then $I = \mathbb{R}$, and the solution u scatters in the sense (1.8).

Remark 1.1. (i) *We remark that the balance between the bounds provided by Lemma 4.1 and the bound required by Theorem 1.3 by making use of the double Duhamel formula is the source of our constraint to dimensions* $d \geqslant 9$. *More precisely, as we will see in the below,* (4.33) *provides the* $L_t^\infty L_x^q$ *bounds for* $q \geqslant 2p$, *while* (4.37) *requires this bound with* $q < \frac{pd}{4}$. *These conditions on* q *impose the restriction* $d \geqslant 9$.

(ii) *Our restriction* $s_c \geqslant 1$ *serves to simplify the analysis for the local theory, which still becomes a bit complicated. However, modifying the argument in the local theory, one may extend Theorem 1.1 to* $s_c \geqslant \frac{1}{2}$ *which enables us to adopt the interaction Morawetz inequality* (*see Lemma 3.1 below*).

(iii) *Finally, we also need that the nonlinearity obeys a certain smoothness condition; more precisely, we ask that* $s_c < 2 + p$ *when* p *is not an even integer. The role of this restriction is to allow us to take* $(s_c - 1)$*-many derivatives of the nonlinearity* $f(u)$. *This is in sharp contrast with NLS, where the restriction for the regularity* $s_c < 1 + p$ *when* p *is not an even integer. The main reason is the Strichartz estimate since there is the smoothing effect for all higher-order nonlinear Schrödinger equations, see Proposition 2 in [29]. This enables us to consider* $s_c < 2 + p$ *for* p *being not even integer in FNLS.*

1.1. **The outline of the proof of Theorem 1.1.** For each $E > 0$, let us define $\Lambda(E)$ to be the quantity

$$\Lambda(E) := \sup \Big\{ S_I(u) : \ u : \ I \times \mathbb{R}^d \to \mathbb{C} \ \text{ such that } \sup_{t \in I} \big\| u \big\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \leqslant E \Big\},$$

where u ranges over all solutions to (1.1) on the spacetime slab $I \times \mathbb{R}^d$ with $\big\|u\big\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \leqslant E$. Thus, $\Lambda : \ [0, +\infty) \to [0, +\infty)$ is a non-decreasing function. Furthermore, from the small data theory, see Proposition 2.2, one has

$$\Lambda(E) \lesssim E \quad \text{for} \quad E \leqslant \eta_0,$$

where $\eta_0 = \eta(d)$ is the threshold from the small data theory.

From the stability theory (see Corollary 2.2 below), we know that Λ is continuous. Thus, there is a unique critical $E_c \in (0, +\infty]$ such that $\Lambda(E) < +\infty$ for $E < E_c$ and $\Lambda(E) = +\infty$ for $E \geqslant E_c$. In particular, if $u : \ I \times \mathbb{R}^d \to \mathbb{C}$ is a maximal-lifespan solution to (1.1) satisfying $\sup_{t \in I} \big\|u\big\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} < E_c$, then u is global and moreover,

$$S_{\mathbb{R}}(u) \leqslant L\big(\big\|u\big\|_{L_t^\infty(\mathbb{R}; \dot{H}^{s_c}(\mathbb{R}^d))} \big).$$

Therefore, the proof of Theorem 1.1 is equivalent to show $E_c = +\infty$. We argue by contradiction. The failure of Theorem 1.1 would imply the existence of very special class of solutions; that is the almost periodicity modulo symmetries:

Definition 1.3. *Let* $s_c \geqslant 1$. *A solution* u *to* (1.1) *with maximal-lifespan* I *is called almost periodic modulo symmetries if* u *is bounded in* $\dot{H}_x^{s_c}(\mathbb{R}^d)$ *and there exist functions* $N(t) : \ I \to \mathbb{R}^+, \ x(t) : \ I \to \mathbb{R}^d$ *and* $C(\eta) : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for all* $t \in I$ *and* $\eta > 0$,

$$\int_{|x - x(t)| \geqslant \frac{C(\eta)}{N(t)}} \big| |\nabla|^{s_c} u(t,x) \big|^2 dx \leqslant \eta \tag{1.11}$$

and

$$\int_{|\xi| \geqslant C(\eta) N(t)} |\xi|^{2s_c} \cdot \big| \hat{u}(t,\xi) \big|^2 d\xi \leqslant \eta. \tag{1.12}$$

We refer to the function $N(t)$ as the frequency scale function for the solution u, to $x(t)$ as the spatial center function, and to $C(\eta)$ as the compactness modules function.

Remark 1.2. *By Ascoli-Arzela Theorem, u is almost periodic modulo symmetries if and only if the set*

$$\Big\{N(t)^{s_c-\frac{d}{2}}u\Big(t, x(t)+\frac{x}{N(t)}\Big),\ t\in I\Big\}$$

falls in a compact set in $\dot{H}^{s_c}_x(\mathbb{R}^d)$. The following are consequences of this statement. If u is almost periodic modulo symmetries, then there exists $c(\eta)>0$ such that

$$\int_{|x-x(t)|\leqslant\frac{c(\eta)}{N(t)}}\big||\nabla|^{s_c}u(t,x)\big|^2dx\leqslant\eta \tag{1.13}$$

and

$$\int_{|\xi|\leqslant c(\eta)N(t)}|\xi|^{2s_c}\cdot|\hat{u}(t,\xi)|^2d\xi\leqslant\eta. \tag{1.14}$$

By the same argument as in [25, 33], we can show that if Theorem 1.1 fails, then we will inevitably encounter at least one of the following three enemies.

Theorem 1.2 (Three enemies, [25, 33])**.** *Suppose $d\geqslant 9$ is such that Theorem 1.1 fails, that is, $E_c<+\infty$. Then there exists a maximal-lifespan solution $u:\ I\times\mathbb{R}^d\to\mathbb{C}$, which is almost periodic modulo symmetries, with $S_I(u)=+\infty$. Furthermore, we can also ensure that the lifespan I and the frequency scale function $N(t):\ I\to\mathbb{R}^+$ satisfy one of the following three scenarios:*

1. (*Finite time blowup*) *Either $|\inf(I)|<+\infty$ or $\sup(I)<+\infty$.*
2. (*Soliton-like solution*) *$I=\mathbb{R}$ and $N(t)=1$ for all $t\in\mathbb{R}$.*
3. (*Low-to-high frequency cascade*) *$I=\mathbb{R}$,*

$$\inf_{t\in\mathbb{R}}N(t)\geqslant 1,\ \text{ and }\ \overline{\lim_{t\to\infty}}\,N(t)=+\infty.$$

In view of this theorem, our goal is to preclude the possibilities of all the scenarios.

We also need the following Duhamel formula, which is important for showing the additional decay and negative regularity in Section 4. This is a robust consequence of almost periodicity modulo symmetries; see, for example, [4, 26, 33].

Lemma 1.1 (No-waste Duhamel formula)**.** *Let $u: I\times\mathbb{R}^d\to\mathbb{C}$ be a maximal-lifespan solution which is almost periodic modulo symmetries. Then, for all $t\in I$, there holds that*

$$\begin{aligned}u(t)&=\lim_{T\nearrow\sup(I)} i\int_t^T e^{i(t-s)\Delta^2}f(u)(s)ds\\&=-\lim_{T\searrow\inf(I)} i\int_T^t e^{i(t-s)\Delta^2}f(u)(s)ds\end{aligned} \tag{1.15}$$

as weak limits in $\dot{H}^{s_c}_x(\mathbb{R}^d)$.

With this lemma in hand, we can deduce that the energy or mass of the finite time blow-up solution is zero and so get a contradiction. We refer to Section 3 for more details.

In view of the no-waste Duhamel formula and noting that the minimal $L_t^\infty \dot{H}_x^{s_c}$-norm blowup solution is localized in both physical and frequency space, we will show that it admits lower regularity.

Theorem 1.3 (Negative regularity in the global case)**.** *Let u be a global solution to* (1.1) *which is almost periodic modulo symmetries in the sense of Theorem 1.2. And assume that* $\inf_{t\in\mathbb{R}} N(t) \geqslant 1$, *then there exists a constant* $\alpha > 0$ *such that for any* $0 < \varepsilon < \alpha$

$$u \in L_t^\infty(\mathbb{R};\ \dot{H}_x^{-\varepsilon}(\mathbb{R}^d)). \tag{1.16}$$

Combining this theorem with interaction Morawetz estimate and interpolation, we will get a contradiction for the global almost periodic solutions in the sense of Theorem 1.2. Thus, we conclude Theorem 1.1. We refer to Section 3 for more details.

The paper is organized as follows. In Section 2, we deal with the local theory for the equation (1.1). In Section 3, we exclude three scenarios in the sense of Theorem 1.2 under the assumption that Theorem 1.3 holds. In Section 4, we show the global solutions which are almost periodic modulus symmetries admit the negative regularity, that is, Theorem 1.3. Hence we conclude the proof of Theorem 1.1. Finally, we show the stability in Appendix.

1.2. **Notations.** Finally, we conclude the introduction by giving some notations which will be used throughout this paper. To simplify the expression of our inequalities, we introduce some symbols $\lesssim, \sim, \ll$. If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leqslant CY$ for some C which may depend on the critical energy E_c but not on any parameter such as η and ρ, and $X \sim Y$ to denote the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \leqslant cY$ for some small constant c which is again allowed to depend on E_c. We use $C \gg 1$ to denote various large finite constants, and $0 < c \ll 1$ to denote various small constants. Any summations over capitalized variables such as M_j are presumed to be dyadic, i.e., these variables range over numbers of the form 2^k for $k \in \mathbb{Z}$. For any $r, 1 \leqslant r \leqslant \infty$, we denote by $\|\cdot\|_r$ the norm in $L^r = L^r(\mathbb{R}^d)$ and by r' the conjugate exponent defined by $\frac{1}{r} + \frac{1}{r'} = 1$. We denote $a\pm$ to be any quantity of the form $a \pm \epsilon$ for any $\epsilon > 0$.

The Fourier transform on $\mathbb{R}^d$ is defined by

$$\widehat{f}(\xi) := \big(2\pi\big)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx,$$

giving rise to the fractional differentiation operators $|\nabla|^s$ and $\langle\nabla\rangle^s$, defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi),\ \widehat{\langle\nabla\rangle^s f}(\xi) := \langle\xi\rangle^s \hat{f}(\xi),$$

where $\langle\xi\rangle := 1 + |\xi|$. This helps us to define the homogeneous and inhomogeneous Sobolev norms

$$\big\|f\big\|_{\dot{H}_x^s(\mathbb{R}^d)} := \big\||\xi|^s \hat{f}\big\|_{L_x^2(\mathbb{R}^d)},\ \big\|f\big\|_{H_x^s(\mathbb{R}^d)} := \big\|\langle\xi\rangle^s \hat{f}\big\|_{L_x^2(\mathbb{R}^d)}.$$

We will also need the Littlewood-Paley projection operators. Specifically, let $\varphi(\xi)$ be a smooth bump function adapted to the ball $|\xi| \leqslant 2$ which equals 1 on the ball $|\xi| \leqslant 1$.

For each dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood-Paley operators

$$\begin{aligned}
\widehat{P_{\leqslant N} f}(\xi) &:= \varphi\Big(\frac{\xi}{N}\Big)\widehat{f}(\xi),\\
\widehat{P_{>N} f}(\xi) &:= \Big(1-\varphi\Big(\frac{\xi}{N}\Big)\Big)\widehat{f}(\xi),\\
\widehat{P_N f}(\xi) &:= \Big(\varphi\Big(\frac{\xi}{N}\Big)-\varphi\Big(\frac{2\xi}{N}\Big)\Big)\widehat{f}(\xi).
\end{aligned}$$

Similarly we can define $P_{<N}$, $P_{\geqslant N}$, and $P_{M<\cdot\leqslant N} = P_{\leqslant N} - P_{\leqslant M}$, whenever M and N are dyadic numbers. We will frequently write $f_{\leqslant N}$ for $P_{\leqslant N} f$ and similarly for the other operators.

The Littlewood-Paley operators commute with derivative operators, the free propagator, and the conjugation operation. They are self-adjoint and bounded on every L^p_x and $\dot{H}^s_x$ space for $1 \leqslant p \leqslant \infty$ and $s \geqslant 0$, moreover, they also obey the following Bernstein estimates

Lemma 1.2 (Bernstein estimates)**.**

$$\begin{aligned}
\big\|P_{\geqslant N} f\big\|_{L^p} &\lesssim N^{-s}\big\||\nabla|^s P_{\geqslant N} f\big\|_{L^p},\\
\big\||\nabla|^s P_{\leqslant N} f\big\|_{L^p} &\lesssim N^{s}\big\|P_{\leqslant N} f\big\|_{L^p},\\
\big\||\nabla|^{\pm s} P_N f\big\|_{L^p} &\sim N^{\pm s}\big\|P_N f\big\|_{L^p},\\
\big\|P_{\leqslant N} f\big\|_{L^q} &\lesssim N^{\frac{d}{p}-\frac{d}{q}}\big\|P_{\leqslant N} f\big\|_{L^p},\\
\big\|P_N f\big\|_{L^q} &\lesssim N^{\frac{d}{p}-\frac{d}{q}}\big\|P_N f\big\|_{L^p},
\end{aligned}$$

where $s \geqslant 0$ and $1 \leqslant p \leqslant q \leqslant \infty$.

2. Preliminaries

2.1. **Strichartz estimate and nonlinear estimates.** In this section, we consider the Cauchy problem for fourth-order Schrödinger equation

$$\begin{cases} iu_t + \Delta^2 u - f(u) = 0,\\ u(0) = u_0. \end{cases} \tag{2.1}$$

The integral equation for the Cauchy problem (2.1) can be written as

$$u(t,x) = e^{i(t-t_0)\Delta^2}u(t_0) - i\int_{t_0}^{t} e^{i(t-s)\Delta^2} f(u(s))ds. \tag{2.2}$$

The biharmonic Schrödinger semigroup is defined for any tempered distribution g by

$$e^{it\Delta^2} g = \mathcal{F}^{-1} e^{it|\xi|^4}\mathcal{F} g.$$

Now we recall the dispersive estimate for the biharmonic Schrödinger operator.

Lemma 2.1 (Dispersive estimate, [1])**.** *Let $2 \leqslant q \leqslant +\infty$. Then, we have the following dispersive estimate*

$$\big\|e^{it\Delta^2} f\big\|_{L^q_x(\mathbb{R}^d)} \leqslant C|t|^{-\frac{d}{4}(1-\frac{2}{q})}\|f\|_{L^{q'}_x(\mathbb{R}^d)} \tag{2.3}$$

for all $t \neq 0$ and $2 \leqslant q \leqslant +\infty$, $\frac{1}{q} + \frac{1}{q'} = 1$.

The Strichartz estimates involve the following definitions:

Definition 2.1. *A pair of Lebesgue space exponents (q, r) are called Schrödinger admissible for $\mathbb{R}^{1+d}$, or denote by $(q, r) \in \Lambda_0$ when $q, r \geqslant 2$, $(q, r, d) \neq (2, \infty, 2)$, and*

$$\frac{2}{q} = d\Big(\frac{1}{2} - \frac{1}{r}\Big). \tag{2.4}$$

Definition 2.2. *In addition, a pair of Lebesgue space exponents (γ, ρ) are called biharmonic admissible for $\mathbb{R}^{1+d}$ or denote by $(\gamma, \rho) \in \Lambda_1$ when $\gamma, \rho \geqslant 2$, $(\gamma, \rho, d) \neq (2, \infty, 4)$, and*

$$\frac{4}{\gamma} = d\Big(\frac{1}{2} - \frac{1}{\rho}\Big). \tag{2.5}$$

For a fixed spacetime slab $I \times \mathbb{R}^d$, we define the Strichartz norm

$$\|u\|_{S^0(I)} := \sup_{(q,r)\in\Lambda_1} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)}.$$

We denote $S^0(I)$ to be the closure of all test functions under this norm and write $N^0(I)$ for the dual of $S^0(I)$.

According to the above dispersive estimate, the abstract duality and interpolation argument(see [14]), we have the following Strichartz estimates.

Proposition 2.1 (Strichartz estimates for Fourth-order Schrödinger [27, 31])**.** *Let $s \geqslant 0$, suppose that $u(t, x)$ is a solution on $[0, T]$ to the initial value problem*

$$\begin{cases} (i\partial_t + \Delta^2)u(t, x) = h, & (t, x) \in [0, T] \times \mathbb{R}^d \\ u(0) = u_0(x), \end{cases} \tag{2.6}$$

for some data u_0 and $T > 0$. Then we have the Strichartz estimate, for $(q, r), (a, b) \in \Lambda_0$

$$\big\||\nabla|^s u\big\|_{L^q([0,T];L^r(\mathbb{R}^d))} \lesssim \big\||\nabla|^{s-\frac{2}{q}} u_0\big\|_{L^2(\mathbb{R}^d)} + \big\||\nabla|^{s-\frac{2}{q}-\frac{2}{a}} h\big\|_{L^{a'}([0,T];L^{b'}(\mathbb{R}^d))}, \tag{2.7}$$

and for $(\gamma, \rho), (c, d) \in \Lambda_1$

$$\|u\|_{L^\gamma([0,T];L^\rho(\mathbb{R}^d))} \lesssim \|u_0\|_{L^2(\mathbb{R}^d)} + \|h\|_{L^{c'}([0,T];L^{d'}(\mathbb{R}^d))}. \tag{2.8}$$

In particular, we have

$$\big\||\nabla|^s u\big\|_{S^0(I)} \lesssim \big\||\nabla|^s u_0\big\|_{L^2(\mathbb{R}^d)} + \big\||\nabla|^s h_1\big\|_{N^0(I)} + \big\||\nabla|^{s-1} h_2\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}, \tag{2.9}$$

where $h = h_1 + h_2$ and $I = [0, T]$.

The key feature of such lemma is that the spacetime-norm of the s-derivative of u can be estimated by $(s-1)$-derivative of the forcing term, which is the consequence of smoothing effect for all higher-order nonlinear Schrödinger equations, see Proposition 2 in [29]. This enables us to consider $s < 2 + p$ for p being not even integer. This is in sharp contrast with NLS, where the restriction for the regularity $s < 1 + p$ when p is not an even integer.

Now we give a few nonlinear estimates which will be applied to show the local well-posedness which is the first step to obtain the global time-space estimate that leads to the scattering.

Lemma 2.2. *(i) (Product rule) Let $s \geqslant 0$, and $1 < r, p_j, q_j < \infty$ such that $\frac{1}{r} = \frac{1}{p_i} + \frac{1}{q_i}$ $(i = 1, 2)$. Then, we have*

$$\big\||\nabla|^s(fg)\big\|_{L_x^r(\mathbb{R}^d)} \lesssim \|f\|_{L_x^{p_1}(\mathbb{R}^d)}\big\||\nabla|^s g\big\|_{L_x^{q_1}(\mathbb{R}^d)} + \big\||\nabla|^s f\big\|_{L_x^{p_2}(\mathbb{R}^d)}\|g\|_{L_x^{q_2}(\mathbb{R}^d)}. \tag{2.10}$$

(ii) (C^1 continuous) Assume that $G \in C^1(\mathbb{C})$, $s \in (0, 1]$, $1 < p, p_1, p_2 < +\infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then, we have

$$\big\||\nabla|^s G(u)\big\|_p \lesssim \|G'(u)\|_{p_1}\big\||\nabla|^s u\big\|_{p_2}. \tag{2.11}$$

(iii) (Hölder continuous) Let $G \in C^\alpha(\mathbb{C})$ with $0 < \alpha < 1$. Then, for every $0 < s < \alpha$, $1 < p < +\infty$, $\frac{s}{\alpha} < \sigma < 1$, we have

$$\big\||\nabla|^s G(u)\big\|_p \lesssim \big\||u|^{\alpha - \frac{s}{\sigma}}\big\|_{p_1}\big\||\nabla|^\sigma u\big\|^{\frac{s}{\sigma}}_{\frac{s}{\sigma}p_2}, \tag{2.12}$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $\big(1 - \frac{s}{\alpha\sigma}\big)p_1 > 1$.

Proof. We refer to [3, 38] for the proof. □

As a direct consequence, we obtain the following nonlinear estimate.

Corollary 2.1. *Let $f(u) = |u|^p u$, and let $s \geqslant 0$ if p is an even integer or $0 \leqslant s < 1 + p$ otherwise. Then, we have*

$$\big\||\nabla|^s f(u)\big\|_{L_x^q} \lesssim \big\||\nabla|^s u\big\|_{L_x^{q_1}}\|u\|^p_{L_x^{q_2}}, \tag{2.13}$$

where $\frac{1}{q} = \frac{1}{q_1} + \frac{p}{q_2}$.

We will also make use of the following refinement of the fractional chain rule, which appears in [21]. This will be used in the proof of the perturbation for $s_c \in [1, 2)$.

Lemma 2.3 (Derivatives of differences, [21])**.** *For $0 < s < 1$ and $f(u) = |u|^p u$. Then for $1 < r, r_1, r_2 < +\infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{p}{r_2}$, we have*

$$\big\||\nabla|^s\big[f(u+v) - f(u)\big]\big\|_r \lesssim \big\||\nabla|^s u\big\|_{r_1}\|v\|^p_{r_2} + \big\||\nabla|^s v\big\|_{r_1}\|u + v\|^p_{r_2}. \tag{2.14}$$

Next, we give a nonlinear estimate in [19]. It is used in the proof of Lemma 5.4.

Lemma 2.4 ([19])**.** *Let $G \in C^\alpha(\mathbb{C})$ with $0 < \alpha \leqslant 1.$, and $0 < s < \sigma\alpha < \alpha$. For $1 < q, q_1, q_2, r_1, r_2, r_3 < +\infty$, such that $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$, we have*

$$\begin{aligned}&\Big\||\nabla|^s\Big[\omega \cdot \big(G(u+v) - G(u)\big)\Big]\Big\|_q \\ \lesssim&\big\||\nabla|^s\omega\big\|_{q_1}\|v\|^p_{\alpha q_2} + \|\omega\|_{r_1}\|v\|^{\alpha - \frac{s}{\sigma}}_{(\alpha - \frac{s}{\sigma})r_2}\Big(\big\||\nabla|^\sigma v\big\|_{\frac{s}{\sigma}r_3} + \big\||\nabla|^\sigma u\big\|_{\frac{s}{\sigma}r_3}\Big)^{\frac{s}{\sigma}},\end{aligned} \tag{2.15}$$

where $(1 - \alpha)r_1$, $\big(\alpha - \frac{s}{\sigma}\big)r_2 > 1$.

We remark that one can extend Lemma 2.4 to $G(u) \simeq O\big(|u|^\alpha\big)$ with $\alpha > 1$, which will be used in the proof of (5.27) for $p > 1$.

We will also need the following lemma which is similar to Lemma 2.11. It is useful to the proof of Proposition 4.1 for $p < 1$.

Lemma 2.5 (Nonlinear Bernstein inequality [23]). *Assume that $G \in C^\alpha(\mathbb{C})$ with $0 < \alpha \leqslant 1$. Then, we have*

$$\big\|P_N G(u)\big\|_{L_x^{\frac{q}{\alpha}}(\mathbb{R}^d)} \lesssim N^{-\alpha}\big\|\nabla u\big\|^{\alpha}_{L_x^q(\mathbb{R}^d)} \tag{2.16}$$

for all $1 \leqslant q < +\infty$.

2.2. **Local well-posedness in inhomogeneous space.** Now we can state the following standard local well-posedness result, where we assume that the initial data in the inhomogeneous critical Sobolev space. This assumption simplifies the proof since one can use the $L_t^q L_x^r$-norm with $(q, r) \in \Lambda_1$ as the metric (that is in mass-critical spaces) when we prove the map is a contraction. And this assumption can be removed by using the perturbation results proved in Corollary 2.2 below, see Proposition 2.2.

Theorem 2.1 (Local well-posedness). *Assume $u_0 \in H_x^{s_c}(\mathbb{R}^d)$, and let $s_c \geqslant 1$ if p is an even integer or $s_c \in [1, 2+p)$ otherwise. Then there exists $\eta_0 = \eta_0(d) > 0$ such that if I is a compact interval containing zero such that*

$$\|e^{it\Delta^2}u_0\|_{Z(I)} := \big\||\nabla|^{s_c-1}e^{it\Delta^2}u_0\big\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{(d-2)(p+1)-4}}(I\times\mathbb{R}^d)} \leqslant \eta, \tag{2.17}$$

where $0 < \eta \leqslant \eta_0$, then there exists a unique solution u to (1.1) on $I \times \mathbb{R}^d$. Furthermore, the solution u obeys

$$\|u\|_{Z(I)} \leqslant 2\eta \tag{2.18}$$

$$\big\||\nabla|^{s_c}u\big\|_{S^0(I)} \leqslant 2C\big\||\nabla|^{s_c}u_0\big\|_{L_x^2} + C\eta^{1+p} \tag{2.19}$$

$$\|u\|_{S^0(I)} \leqslant 2C\|u_0\|_{L_x^2}, \tag{2.20}$$

where C is the Strichartz constant as in Proposition 2.1.

Proof. We apply the Banach fixed point argument to prove this lemma. First we define the map

$$\Phi(u(t)) = e^{it\Delta^2}u_0 - i\int_0^t e^{i(t-s)\Delta^2}f(u(s))ds \tag{2.21}$$

on the complete metric space B

$$\begin{aligned} B := \big\{u \in C_t(I; H_x^{s_c}) : \|u\|_{L_t^\infty H_x^{s_c}(I\times\mathbb{R}^d)} \leqslant 2C\|u_0\|_{H_x^{s_c}} + C\eta^{1+p};\\ \|u\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{d(p+1)-4}}(I\times\mathbb{R}^d)} \leqslant 2C\|u_0\|_{L_x^2};\ \|u\|_{Z(I)} \leqslant 2\eta\big\} \end{aligned}$$

with the metric $d(u, v) = \big\|u - v\big\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{d(p+1)-4}}(I\times\mathbb{R}^d)}$.

It suffices to prove that the operator defined by the RHS of (2.21) is a contraction map on B for I. If $u \in B$, then by Strichartz estimate, Corollary 2.1 and (2.17), we have

$$\begin{aligned} \|\Phi(u)\|_{Z(I)} =& \big\||\nabla|^{s_c-1}\Phi(u)\big\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{(d-2)(p+1)-4}}(I\times\mathbb{R}^d)}\\ \leqslant& \|e^{it\Delta^2}u_0\|_{Z(I)} + C\big\||\nabla|^{s_c-1}f(u)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ \leqslant& \eta + C\|u\|^{p+1}_{Z(I)}. \end{aligned}$$

Plugging the assumption $\|u\|_{Z(I)} \leqslant 2\eta$, we see that for $u \in B$,

$$\|\Phi(u)\|_{Z(I)} \leqslant \eta + 8C\eta^{p+1} \leqslant 2\eta$$

provided we take η sufficiently small such that $8C\eta^p \leqslant 1$. Similarly, if $u \in B$, then

$$\begin{aligned}
&\|\Phi(u)\|_{L_t^\infty H_x^{s_c}(I\times\mathbb{R}^d)}\\
\leqslant& C\|u_0\|_{H_x^{s_c}(\mathbb{R}^d)} + C\big\||\nabla|^{s_c-1}f(u)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} + C\|f(u)\|_{L_t^2L_x^{\frac{2d}{d+4}}(I\times\mathbb{R}^d)}\\
\leqslant& C\|u_0\|_{H_x^{s_c}(\mathbb{R}^d)} + C\|u\|_{Z(I)}\|u\|^p_{L_t^{2(p+1)}L_x^{\frac{dp(p+1)}{2(p+2)}}(I\times\mathbb{R}^d)} + C\|u\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{d(p+1)-4}}(I\times\mathbb{R}^d)}\|u\|^p_{Z(I)}\\
\leqslant& C\|u_0\|_{H_x^{s_c}(\mathbb{R}^d)} + C(2\eta)^{p+1} + 2C\|u_0\|_{L_x^2}(2\eta)^p\\
\leqslant& 2C\|u_0\|_{H_x^{s_c}(\mathbb{R}^d)} + C\eta^{1+p},
\end{aligned}$$

and

$$\begin{aligned}
\big\|\Phi(u)\big\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{d(p+1)-4}}(I\times\mathbb{R}^d)} \leqslant& C\|u_0\|_{L_x^2(\mathbb{R}^d)} + C\big\|f(u)\big\|_{L_t^2L_x^{\frac{2d}{d+4}}(I\times\mathbb{R}^d)}\\
\leqslant& C\|u_0\|_{L_x^2(\mathbb{R}^d)} + C\|u\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{d(p+1)-4}}(I\times\mathbb{R}^d)}\|u\|^p_{Z(I)}\\
\leqslant& C\|u_0\|_{L_x^2(\mathbb{R}^d)} + C^2\|u_0\|_{L_x^2(\mathbb{R}^d)}(2\eta)^p\\
\leqslant& 2C\|u_0\|_{L_x^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence $\Phi(u) \in B$ for $u \in B$. That is, the functional Φ maps the set B back to itself.

On the other hand, by the same argument as before, we have for $u, v \in B$,

$$\begin{aligned}
d\big(\Phi(u),\Phi(v)\big) \leqslant& C\big\|f(u)-f(v)\big\|_{L_t^2L_x^{\frac{2d}{d+4}}(I\times\mathbb{R}^d)}\\
\leqslant& C\|u-v\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{d(p+1)-4}}(I\times\mathbb{R}^d)}\|(u,v)\|^p_{Z(I)}\\
\leqslant& C(4\eta)^p d(u,v)
\end{aligned}$$

which allows us to derive

$$d\big(\Phi(u),\Phi(v)\big) \leqslant \frac{1}{2}d(u,v),$$

by taking η small such that

$$C(4\eta)^p \leqslant \frac{1}{2}.$$

A standard fixed point argument gives a unique solution u of (1.1) on $I \times \mathbb{R}^d$ which satisfies the bound (2.18). The bounds (2.19) and (2.20) follow from another application of the Strichartz estimate. □

2.3. **Perturbation.** Closely related to the continuous dependence on the data, an essential tool for concentration compactness arguments is the perturbation theory. And we will show this perturbation theory in Appendix.

Lemma 2.6 (Perturbation Lemma). *Let $s_c \geqslant 1$. Assume in addition that $s_c < 2+p$ if p is not an even integer. Let I be a compact time interval and u, $\tilde{u}$ satisfy*

$$\begin{aligned}
(i\partial_t + \Delta^2)u =& -f(u) + eq(u)\\
(i\partial_t + \Delta^2)\tilde{u} =& -f(\tilde{u}) + eq(\tilde{u})
\end{aligned}$$

for some function $eq(u), eq(\tilde{u})$, and $f(u) = |u|^p u$. Assume that for some constants $M, E > 0$, we have

$$\|u\|_{L_t^\infty(I;\dot{H}_x^{s_c}(\mathbb{R}^d))} + \|\tilde{u}\|_{L_t^\infty(I;\dot{H}_x^{s_c}(\mathbb{R}^d))} \leqslant E, \tag{2.22}$$

$$S_I(\tilde{u}) \leqslant M, \tag{2.23}$$

Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u_0 - \tilde{u}_0\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \leqslant \varepsilon, \tag{2.24}$$

where $0 < \varepsilon < \varepsilon_1(M, E)$ is a small constant. Assume also that we have smallness conditions

$$\big\||\nabla|^{s_c-1}\big(eq(u), eq(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \leqslant \varepsilon, \tag{2.25}$$

where ε is as above.

Then we conclude that

$$\begin{aligned} S_I(u - \tilde{u}) &\leqslant C(M, E)\varepsilon^{c_1} \\ \big\||\nabla|^{s_c}(u - \tilde{u})\big\|_{S^0(I)} &\leqslant C(M, E)\varepsilon^{c_2} \\ \big\||\nabla|^{s_c}u\big\|_{S^0(I)} &\leqslant C(M, E), \end{aligned} \tag{2.26}$$

where c_1, c_2 are positive constants that depend on d, p, E and M.

2.4. **Local well-posedness in homogenous space and stability.** As stated in the subsection 2.2, the assumption that the initial data in the inhomogeneous critical Sobolev space can be removed by the perturbation results. Now we give a detail proof.

Proposition 2.2 (Local well-posedness in homogenous space)**.** *Assume that $s_c \geqslant 1$ if p is an even integer or $1 \leqslant s_c < 2 + p$ otherwise. Let $u_0 \in \dot{H}_x^{s_c}(\mathbb{R}^d)$. Then, if I is a compact interval containing zero such that*

$$\|e^{it\Delta^2}u_0\|_{Z(I)} := \big\||\nabla|^{s_c-1}e^{it\Delta^2}u_0\big\|_{L_t^{2(p+1)}L_x^{\frac{2d(p+1)}{(d-2)(p+1)-4}}(I\times\mathbb{R}^d)} \leqslant \frac{\eta}{2}, \tag{2.27}$$

where η is as in Theorem 2.1, then there exists a unique solution u to (1.1) on $I \times \mathbb{R}^d$. Furthermore, the solution u satisfies the bounds

$$\|u\|_{Z(I)} \leqslant 2\eta \tag{2.28}$$

$$\big\||\nabla|^{s_c}u\big\|_{S^0(I)} \leqslant 2C\big\||\nabla|^{s_c}u_0\big\|_{L_x^2} + C\eta^{1+p}, \tag{2.29}$$

where C is the Strichartz constant as in Proposition 2.1.

In particular, if $\|u_0\|_{\dot{H}_x^{s_c}(\mathbb{R}^d)} \leqslant \frac{\eta}{2}$, then the solution u is global and scatters.

Proof. Since $H^{s_c}(\mathbb{R}^d)$ is dense in $\dot{H}^{s_c}(\mathbb{R}^d)$, we know that for any $u_0 \in \dot{H}^{s_c}(\mathbb{R}^d)$, there exists a sequence $\{u_n(0)\} \subset H^{s_c}(\mathbb{R}^d)$ such that

$$\|u_n(0) - u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)} \to 0, \text{ as } \quad n \to +\infty.$$

Hence, $\forall\, \varepsilon > 0,\ \exists\, N > 0,\ s.t.\ \forall\, n > N,$

$$\|u_n(0) - u_0\|_{\dot{H}^{s_c}(\mathbb{R}^d)} < \varepsilon.$$

By Strichartz estimate and (2.27), we get for $2C\varepsilon < \eta$ and $n > N$,

$$\big\|e^{it\Delta^2}u_n(0)\big\|_{Z(I)} \leqslant C\|u_n(0)-u_0\|_{\dot H^{s_c}(\mathbb{R}^d)} + \big\|e^{it\Delta^2}u_0\big\|_{Z(I)} \leqslant C\varepsilon + \frac{\eta}{2} \leqslant \eta.$$

This together with $u_n(0) \in H^{s_c}(\mathbb{R}^d)$, and Theorem 2.1 yield that there exists a unique solution $u_n(t,x):\ I\times\mathbb{R}^d \to \mathbb{C}$ to (1.1) with initial data $u_n(0)$ obeying (2.18)-(2.20). In particular, it satisfies

$$\big\||\nabla|^{s_c}u_n\big\|_{S^0(I\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c}u_n(0)\big\|_{L^2_x} + \eta^{1+p} \lesssim \big\||\nabla|^{s_c}u_0\big\|_{L^2_x} + \eta^{1+p} + \varepsilon. \tag{2.30}$$

Next we use Lemma 2.6 to show the solution sequence $\{u_n(t,x)\}$ is Cauchy in $S^{s_c}(I)$, where $\|u\|_{S^{s_c}(I)} := \big\||\nabla|^{s_c}u\big\|_{S^0(I)}$. In fact, it follows from Lemma 2.6 if we set $\tilde u = u_m,\ u = u_n$, and $eq(u) = eq(\tilde u) = 0$. Thus, by (2.26), we get

$$\big\||\nabla|^{s_c}(u_n-u_m)\big\|_{S^0(I)} \leqslant C(E,M)\varepsilon,$$

which means $\{u_n(t,x)\}$ is Cauchy in $S^{s_c}(I)$. And so it convergent to a solution $u(t,x)$ with initial data $u(0,x)=u_0$ obeying $|\nabla|^{s_c}u \in S^0(I)$. □

Using the Theorem 2.1 and Lemma 2.6 as well as their proof, one easily derives the following local theory for (1.1). We refer the author to Pausader [31] for the special energy-critical case ($s_c = 2$).

Theorem 2.2. *Let $s_c \geqslant 1$. Assume in addition that $s_c < 2+p$ if p is not an even integer. Then, given $u_0 \in \dot H^{s_c}_x(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution $u : I\times\mathbb{R}^d \to \mathbb{C}$ to* (1.1) *with initial data $u(t_0) = u_0$. This solution also has the following properties:*

1. (*Local existence*) *I is an open neighborhood of t_0.*
2. (*Blowup criterion*) *If $\sup(I)$ is finite, then u blows up forward in time in the sense of Definition 1.2. If $\inf(I)$ is finite, then u blows up backward in time.*
3. (*Scattering*) *If $\sup(I) = +\infty$ and u does not blow up forward in time, then u scatters forward in time in the sense* (1.8). *Conversely, given $v_+ \in \dot H^{s_c}(\mathbb{R}^d)$, there is a unique solution to* (1.1) *in a neighborhood of infinity so that* (1.8) *holds.*

It is easy to show the following stability result by Proposition 2.2 and Lemma 2.6 as well as their proof.

Corollary 2.2 (stability)**.** *Assume that $s_c \geqslant 1$ if p is an even integer or $1 \leqslant s_c < 2+p$ otherwise. Let I be a compact time interval containing zero and $\tilde u$ be an near solution to* (1.1) *on $I\times\mathbb{R}^d$ in the sense that*

$$i\tilde u_t + \Delta\tilde u - f(\tilde u) + e = 0$$

for some function e. Assume that for some constants $M, E > 0$, we have

$$\|\tilde u\|_{L^\infty_t(I;\dot H^{s_c}_x(\mathbb{R}^d))} \leqslant E, \tag{2.31}$$

$$S_I(\tilde u) \leqslant M. \tag{2.32}$$

Let $u_0 \in \dot H^{s_c}_x(\mathbb{R}^d)$ and assume the smallness conditions

$$\|u_0 - \tilde u_0\|_{\dot H^{s_c}_x(\mathbb{R}^d)} + \big\||\nabla|^{s_c-1}e\big\|_{L^2_tL^{\frac{2d}{d+2}}_x(I\times\mathbb{R}^d)} \leqslant \varepsilon \tag{2.33}$$

where $0<\varepsilon<\epsilon_1=\epsilon_1(M,E)$ is a small constant. Then there exists a unique solution $u: I\times\mathbb{R}^d\to\mathbb{C}$ to (1.1) with initial data u_0 at time $t=0$ obeying

$$S_I(u-\tilde{u}) \leqslant C(M,E)\varepsilon^{c_1} \tag{2.34}$$

$$\big\||\nabla|^{s_c}(u-\tilde{u})\big\|_{S^0(I)} \leqslant C(M,E)\varepsilon^{c_2}, \tag{2.35}$$

$$\big\||\nabla|^{s_c}u\big\|_{S^0(I)} \leqslant C(M,E). \tag{2.36}$$

where $c_1,\ c_2$ are positive constants that depend on d,p,E and M.

3. Extinction of three scenarios

In this section, we preclude three scenarios in the sense of Theorem 1.2 under the assumption that Theorem 1.3 holds. We will prove Theorem 1.3 in the next section. First, we preclude the finite time blowup solution by making use of No-waste Duhamel formula.

3.1. **The finite blowup solution.** We argue by contradiction. Assume that there exists a solution $u: I\times\mathbb{R}^d\to\mathbb{C}$ which is a finite time blowup in the sense of Theorem 1.2. Assume also $T:=\sup(I)<+\infty$, then, we have by (1.10) and Sobolev embedding

$$\|u\|_{L_t^\infty L_x^{\frac{pd}{4}}(I\times\mathbb{R}^d)} \lesssim \|u\|_{L_t^\infty \dot{H}^{s_c}(I\times\mathbb{R}^d)} \lesssim 1. \tag{3.1}$$

First, we consider the energy-subcritical and energy-critical case.

Case 1: $1\leqslant s_c\leqslant 2$. Using Strichartz estimate, Sobolev embedding, (3.1), (1.15) and Hölder's inequality, we have

$$\begin{aligned}
\big\||\nabla|^{s_c-2}u(t)\big\|_{L_x^2} \leqslant& \Big\|\int_t^T e^{i(t-s)\Delta^2}|\nabla|^{s_c-2}f(u(s))ds\Big\|_{L_x^2}\\
\lesssim& \big\||\nabla|^{s_c-2}f(u(s))\big\|_{L_t^2 L_x^{\frac{2d}{d+4}}([t,T)\times\mathbb{R}^d)}\\
\lesssim& (T-t)^{\frac12}\|f(u)\|_{L_t^\infty L_x^{\frac{pd}{4(p+1)}}([t,T)\times\mathbb{R}^d)}\\
\lesssim& (T-t)^{\frac12}\|u\|^{p+1}_{L_t^\infty L_x^{\frac{pd}{4}}([t,T)\times\mathbb{R}^d)}\\
\lesssim& (T-t)^{\frac12}.
\end{aligned}$$

Interpolating this with $u\in L_t^\infty\dot{H}^{s_c}([0,T)\times\mathbb{R}^d)$, we deduce that

$$\|u(t)\|_{L_x^2}\lesssim (T-t)^{\frac{s_c}{4}}\to 0,\ \text{as } t\to T$$

which shows that $u\in L_t^\infty L_x^2([0,T)\times\mathbb{R}^d)$ and also $u\equiv 0$ by the mass conservation. This contradicts with the fact that u is a blowup solution.

Next, we consider the energy-supercritical case. Using the assumption (1.10) and Sobolev embedding, we have

$$\|u\|_{L_t^\infty L_x^{\frac{pd}{4}}(I\times\mathbb{R}^d)} \lesssim \big\||\nabla|^{s_c-2}u\big\|_{L_t^\infty L_x^{\frac{2d}{d-4}}(I\times\mathbb{R}^d)} \lesssim \|u\|_{L_t^\infty\dot{H}^{s_c}(I\times\mathbb{R}^d)}\lesssim 1. \tag{3.2}$$

Case 2: $s_c \in (2,4]$. Combining (3.2) with No waste Duhamel formula (1.15), Strichartz estimate, Hölder's inequality and Corollary 2.1, one has

$$
\begin{aligned}
\big\||\nabla|^{s_c-2}u(t)\big\|_{L_x^2} \leqslant & \Big\|\int_t^T e^{i(t-s)\Delta^2}|\nabla|^{s_c-2}f(u(s))ds\Big\|_{L_x^2} \\
\lesssim & \big\||\nabla|^{s_c-2}f(u(s))\big\|_{L_t^2 L_x^{\frac{2d}{d+4}}([t,T)\times\mathbb{R}^d)} \\
\lesssim & (T-t)^{\frac12}\big\||\nabla|^{s_c-2}u\big\|_{L_t^\infty L_x^{\frac{2d}{d-4}}([t,T)\times\mathbb{R}^d)}\|u\|^p_{L_t^\infty L_x^{\frac{pd}{4}}([t,T)\times\mathbb{R}^d)} \\
\lesssim & (T-t)^{\frac12}.
\end{aligned}
\tag{3.3}
$$

Interpolating this with (1.10), we derive that

$$E(u_0)=E(u(t))\to 0, \quad \text{as} \quad t\to T,$$

which implies that $u\equiv 0$. This contradicts with the fact that u is a blowup solution.

Case 3: $s_c \in (4,6]$**.** It follows from (3.3) that $u\in L_t^\infty([0,T);\dot H_x^{s_c-2}(\mathbb{R}^d))$. Using No waste Duhamel formula (1.15), Strichartz estimate, Hölder's inequality and fractional chain rule, we obtain

$$
\begin{aligned}
\big\||\nabla|^{s_c-4}u(t)\big\|_{L_x^2} \leqslant & \Big\|\int_t^T e^{i(t-s)\Delta^2}|\nabla|^{s_c-4}f(u(s))ds\Big\|_{L_x^2} \\
\lesssim & \big\||\nabla|^{s_c-4}f(u(s))\big\|_{L_t^2 L_x^{\frac{2d}{d+4}}([t,T)\times\mathbb{R}^d)} \\
\lesssim & (T-t)^{\frac12}\big\||\nabla|^{s_c-4}u\big\|_{L_t^\infty L_x^{\frac{2d}{d-4}}([t,T)\times\mathbb{R}^d)}\|u\|^p_{L_t^\infty L_x^{\frac{pd}{4}}([t,T)\times\mathbb{R}^d)} \\
\lesssim & (T-t)^{\frac12},
\end{aligned}
\tag{3.4}
$$

Interpolating this with (1.10) again, we also deduce that

$$E(u_0)=E(u(t))\to 0, \quad \text{as} \quad t\to T.$$

This contradicts with the fact that u is a blowup solution.

Case 4: $s_c \in (6,+\infty)$**.** We can iterate the argument presented above to obtain the contradiction.

Hence, we exclude the finite time blowup solution in the sense of Theorem 1.2.

3.2. **The soliton-like solution.** Next, we adopt the interaction Morawetz estimate to kill the soliton-like solution.

We argue by contradiction. Assume that there exists a solution $u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$ which is a soliton-like solution in the sense of Theorem 1.2. Assume also Theorem 1.3 holds. In particular, we have

$$u(t,x)\in L_t^\infty(\mathbb{R};L_x^2(\mathbb{R}^d)). \tag{3.5}$$

Therefore, the solution u satisfies the following interaction Morawetz estimate.

Lemma 3.1 (Interaction Morawetz estimate, [28, 32])**.** *Assume that $d\geqslant 7$. Let $u:\mathbb{R}\times\mathbb{R}^d\to\mathbb{C}$ be the solution to* (1.1)*, and $u\in L_t^\infty(\mathbb{R};H_x^{\frac12}(\mathbb{R}^d))$. Then, for any compact*

interval $I \subset \mathbb{R}$, we have

$$\int_I \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(t,x)|^2|u(t,y)|^2}{|x-y|^5} dxdydt \lesssim \|u\|^2_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \big\||\nabla_x|^{\frac12}u\big\|^2_{L_t^\infty L_x^2(I\times\mathbb{R}^d)} \lesssim 1. \tag{3.6}$$

From (3.6), we know that

$$\big\||\nabla|^{-\frac{d-5}{2}}(|u|^2)\big\|_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim 1.$$

And so, it follows from [32] that

$$\begin{aligned}\big\||\nabla|^{-\frac{d-5}{4}}u\big\|_{L^4_{t,x}(I\times\mathbb{R}^d)} \simeq& \Big\|\Big(\sum_{N\in 2^{\mathbb{Z}}} N^{-\frac{d-5}{2}}|P_N u|^2\Big)^{\frac12}\Big\|_{L^4_{t,x}(I\times\mathbb{R}^d)} \\ \lesssim& \big\||\nabla|^{-\frac{d-5}{2}}(|u|^2)\big\|^{\frac12}_{L^2_{t,x}(I\times\mathbb{R}^d)} \lesssim 1.\end{aligned} \tag{3.7}$$

Interpolating this with $u \in L_t^\infty(\mathbb{R}; \dot H_x^1(\mathbb{R}^d))$, we obtain for all compact time interval $I \subset \mathbb{R}$

$$\|u\|_{L_t^{d-1}L_x^{\frac{2(d-1)}{d-3}}(I\times\mathbb{R}^d)} \lesssim 1. \tag{3.8}$$

Now we claim that

$$\|u\|_{L_x^{\frac{2(d-1)}{d-3}}(\mathbb{R}^d)} \gtrsim 1, \quad \text{uniformly for} \quad t\in\mathbb{R}. \tag{3.9}$$

If this claim holds, then we derive a contradiction by taking the length of the interval I to be sufficiently large.

Hence it suffices to prove the claim (3.9). We argue by contradiction. Suppose that the claim fails, then there exists a time sequence $\{t_n\}$ such that $u(t_n)$ converges to zero in $L_x^{\frac{2(d-1)}{d-3}}$. On the other hand, $u(t_n)$ converges weakly to zero in $\dot H^{s_c}(\mathbb{R}^d)$ since $u(t)$ is uniformly bounded in $\dot H^{s_c}(\mathbb{R}^d)$. This contradicts with the fact that the orbit of u is precompact in $\dot H^{s_c}(\mathbb{R}^d)$ and u is not identically zero.

And so the claim holds. This completes the proof of excluding the soliton-like solution in the sense of Theorem 1.2.

3.3. **Low to high frequency cascade.** Finally, we turn to exclude the low to high frequency cascade solution.

We argue by contradiction. Assume that there exists a solution $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ which is a low to high frequency cascade solution in the sense of Theorem 1.2. Assume also that Theorem 1.3 holds. In particular, there exists $\varepsilon > 0$ such that

$$u(t,x) \in L_t^\infty(\mathbb{R}; \dot H_x^{-\varepsilon}(\mathbb{R}^d)). \tag{3.10}$$

From $\overline{\lim_{t\to+\infty}} N(t) = +\infty$, we can find a time sequence $\{t_n\}$ such that

$$\lim_{n\to+\infty} N(t_n) = +\infty. \tag{3.11}$$

Using Bernstein's inequality, interpolation, the compactness (1.14), the hypothesis (3.10), and the assumption (1.10), we have

$$\begin{aligned}\|u(t_n,x)\|_{L^2_x} \leqslant&\|P_{\leqslant c(\eta)N(t_n)}u\|_{L^2_x}+\|P_{\geqslant c(\eta)N(t_n)}u\|_{L^2_x}\\ \lesssim&\|u\|_{\dot H^{-\varepsilon}}^{\frac{s_c}{s_c+\varepsilon}}\big\|P_{\leqslant c(\eta)N(t_n)}u\big\|_{\dot H^{s_c}_x}^{\frac{\varepsilon}{s_c+\varepsilon}}+\big(c(\eta)N(t_n)\big)^{-s_c}\|u\|_{\dot H^{s_c}_x}\\ \lesssim&\eta^{\frac{\varepsilon}{s_c+\varepsilon}}+\big(c(\eta)N(t_n)\big)^{-s_c},\end{aligned} \tag{3.12}$$

Taking η small, and then n large, we have by (3.11) and (3.12)

$$M(u_0)=M(u(t_n))\to 0,\ \text{as}\ n\to\infty,$$

which implies that $u\equiv 0$. This contradicts with the fact that u is a blowup solution. Therefore, we preclude the low to high frequency cascade solution in the sense of Theorem 1.2.

In sum, it reduces to prove Theorem 1.3.

4. Negative regularity

As stated in Section 3, it remains to show Theorem 1.3. That is, we need to prove that the global solutions to (1.1) which are almost periodic modulo symmetries enjoy the negative regularity. We will divide two steps to prove it. First, we show additional decay for the soliton-like and frequency-cascade solutions in the sense of Theorem 1.2. And then, this together with the double Duhamel trick yields the negative regularity for the soliton-like and frequency-cascade solutions.

4.1. **Additional Decay.**

We first consider the energy-supercritical case.

Proposition 4.1 (Additional decay I, energy-supercritical)**.** *Let $d\geqslant 9$ and $s_c>2$. Assume in addition that $s_c<2+p$ if p is not an even integer. And let u be a global solution to* (1.1) *that is almost periodic modulo symmetries. In particular,*

$$\|u\|_{L^\infty_t(\mathbb{R};\dot H^{s_c}_x(\mathbb{R}^d))}<+\infty. \tag{4.1}$$

And assume that $\inf\limits_{t\in\mathbb{R}}N(t)\geqslant 1$. Then, we have

$$u\in L^\infty_t(\mathbb{R};L^q_x(\mathbb{R}^d)),\quad q\in\Big(\frac{2d}{d-4},\frac{d}{4}p\Big]. \tag{4.2}$$

Remark 4.1. (*i*) *It is easy to see that we have by Sobolev embedding and* (4.1)

$$u\in L^\infty_tL_x^{\frac{pd}{4}}(\mathbb{R}\times\mathbb{R}^d).$$

(*ii*) (4.2) *can be reduced to show that there exists $\alpha>0$ and $N_0\in 2^{\mathbb{Z}}$ such that for all dyadic number $N\leqslant N_0$*

$$\|u_N\|_{L^\infty_tL^q_x(\mathbb{R}^d)}\lesssim N^\alpha,\quad q\in\Big(\frac{2d}{d-4},\frac{d}{4}p\Big].$$

In fact, we have by Bernstein's inequality and (4.1)

$$\begin{aligned}\|u\|_{L^q_x(\mathbb{R}^d)} \lesssim & \sum_{N\leqslant N_0} \|u_N\|_{L^q_x} + \big\|P_{\geqslant N_0}u\big\|_{L^q_x}\\ \lesssim & \sum_{N\leqslant N_0} N^\alpha + \big\||\nabla|^{\frac{d}{2}-\frac{d}{q}}P_{\geqslant N_0}u\big\|_{L^2_x}\\ \lesssim & N_0^\alpha + N_0^{\frac{d}{2}-\frac{d}{q}-s_c}\big\||\nabla|^{s_c}u\big\|_{L^2_x} < +\infty.\end{aligned}$$

The proof of Proposition 4.1: From (1.14), we know that

$$\|u_{\leqslant c(\eta)N(t)}\|_{\dot H^{s_c}} \leqslant \eta.$$

Combining this with $\inf\limits_{t\in\mathbb{R}} N(t) \geqslant 1$, we deduce that if we take N_0 such that $N_0 \leqslant c(\eta)$, then

$$\|u_{\leqslant N_0}\|_{\dot H^{s_c}} \leqslant \eta. \tag{4.3}$$

Now we define $A_q(N)$ by

$$A_q(N) = N^{\frac{d}{q}-\frac{4}{p}}\|u_N\|_{L^\infty_t(\mathbb{R};L^q_x(\mathbb{R}^d))},\ q > \frac{2d}{d-4}, \tag{4.4}$$

It is easy to see that $A_q(N) \lesssim 1$ by Bernstein's inequality and (4.1).

We first consider that p is an even integer.

Case 1: p **even.** We claim that $A_q(N)$ satisfies the following recurrence formula

$$\begin{aligned}A_q(N) \lesssim & \Big(\frac{N}{N_0}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}} + \eta^p \sum_{\frac{N}{10p}\leqslant M\leqslant N_0} \Big(\frac{N}{M}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}} A_q(M)\\ & + \eta^p \sum_{M\leqslant \frac{N}{10p}} \Big(\frac{M}{N}\Big)^{-\frac{d}{2}+4+\frac{d}{q}-} A_q(M)\end{aligned} \tag{4.5}$$

for any $q > \frac{2d}{d-4}$. Note that $d-4-\frac{4}{p}-\frac{d}{q},\ -\frac{d}{2}+4+\frac{d}{q}- > 0$ whenever $q \in \Big(\frac{2d}{d-4}, \frac{2d}{d-8}\Big)$.

We postpone the proof of this claim. And we recall a acausal Gronwall inequality.

Lemma 4.1 (Acausal Gronwall inequality [22])**.** *Given* $\eta, C, \gamma, \gamma' > 0$*, let* $\{x_k\}_{k\geqslant 0}$ *be a bounded nonnegative sequence obeying*

$$x_k \leqslant C2^{-\gamma k} + \eta\sum_{l=0}^{k-1} 2^{-\gamma(k-l)}x_l + \eta\sum_{l\geqslant k} 2^{-\gamma'(l-k)}x_l \tag{4.6}$$

for all $k \geqslant 0$*. If* $\eta \leqslant \frac14\min\{1-2^{-\gamma}, 1-2^{-\gamma'}, 1-2^{\rho-\gamma}\}$ *for some* $0<\rho<\gamma$*, then*

$$x_k \leqslant (4C + \|x\|_{l^\infty})2^{-\rho k}. \tag{4.7}$$

Now we use the claim (4.5) to prove Proposition 4.1 for p being an even integer. Applying Lemma 4.1 with $x_k = A_q(2^{-k}N_0)$, we obtain by (4.5)

$$x_k \leqslant C2^{-k\left(d-4-\frac{4}{p}-\frac{d}{q}\right)} + C\eta^p\sum_{l=0}^{k} 2^{-(k-l)\left(d-4-\frac{4}{p}-\frac{d}{q}\right)}x_l + C\eta^p\sum_{l>k} 2^{-(l-k)\left(-\frac{d}{2}+4+\frac{d}{q}-\right)}x_l.$$

Then $x_k \lesssim 2^{-k\rho},\ 0<\rho<d-4-\frac{2}{p}-\frac{d}{q}$, that is,

$$A_q(N) \lesssim N^{(d-4-\frac{4}{p}-\frac{d}{q})-},\ q\in\Big(\frac{2d}{d-4},\frac{2d}{d-8}\Big)$$

which means for $N \leqslant N_0$

$$\|u_N\|_{L^q_x} \lesssim N^{\left(d-4-\frac{2d}{q}\right)-},\ q\in\Big(\frac{2d}{d-4},\frac{2d}{d-8}\Big). \tag{4.8}$$

This together with Remark 4.1 (ii) yields that

$$u\in L^\infty_t(\mathbb{R};L^q_x(\mathbb{R}^d)),\ q\in\Big(\frac{2d}{d-4},\min\Big\{\frac{2d}{d-8},\frac{dp}{4}\Big\}\Big).$$

Interpolating this with $u\in L^\infty_t L_x^{\frac{pd}{4}}(\mathbb{R}\times\mathbb{R}^d)$, we conclude Proposition 4.1 for p being an even integer.

Therefore, it suffices to prove the claim (4.5). By time-translation symmetry, we only need to estimate (4.4) at $t=0$. Using No waste Duhamel formula (1.15), Bernstein's inequality, dispersive estimate (2.3), we obtain for all $q>\frac{2d}{d-4}$

$$\begin{aligned}\|u_n(0)\|_{L^q_x} \leqslant& \Big\|\int_0^{+\infty} e^{it\Delta^2}P_Nf(u)(t)dt\Big\|_{L^q_x}\\ \lesssim& N^{d\left(\frac12-\frac1q\right)}\int_0^{N^{-4}}\Big\|e^{it\Delta^2}P_Nf(u)\Big\|_{L^2_x}dt+\int_{N^{-4}}^{+\infty}t^{-d\left(\frac14-\frac{1}{2q}\right)}\|P_Nf(u)\|_{L^{q'}_x}dt\\ \lesssim& N^{d-4-\frac{2d}{q}}\|P_Nf(u)\|_{L^\infty_tL^{q'}_x}.\end{aligned}$$

Thus

$$A_q(N)\lesssim N^{d-4-\frac4p-\frac dq}\|P_Nf(u)\|_{L^\infty_tL^{q'}_x},\ q>\frac{2d}{d-4}. \tag{4.9}$$

Decomposing u by

$$u=u_{>N_0}+u_{\leqslant N_0}=u_{>N_0}+u_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}+u_{<\frac{N}{10p}}$$

and using the fact that p is an even integer, we can write $P_Nf(u)$ by

$$P_Nf(u)=P_N\Big[\varnothing\Big(u_{>N_0}\sum_{k=0}^{p}u^k_{>N_0}u^{p-k}_{\leqslant N_0}\Big)+\varnothing\Big(\sum_{k=0}^{p}u^k_{<\frac{N}{10p}}u^{p+1-k}_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\Big)\Big]. \tag{4.10}$$

Here we use the notation $\varnothing(X)$ to denote a quantity that resembles X, that is, a finite linear combination of terms that look like those in X, but possibly with some factors replaced by their complex conjugates and/or restricted to various frequencies.

We first consider the terms which contain at least one factor of $u_{>N_0}$. By Hölder's inequality, Bernstein's inequality, Sobolev embedding: $\dot H^{s_c}_x(\mathbb{R}^d)\hookrightarrow L_x^{\frac{pd}{4}}(\mathbb{R}^d)$ and the

assumption (4.1), we get

$$
\begin{aligned}
\big\|P_N\varnothing\big(u_{>N_0}\cdot u^p\big)\big\|_{L_t^\infty L_x^{q'}} \lesssim& \|u_{>N_0}\|_{L_t^\infty L_x^r}\|u\|^p_{L_t^\infty L_x^{\frac{pd}{4}}}\\
\lesssim& N_0^{-d+4+\frac{4}{p}+\frac{d}{q}}\|u\|^{p+1}_{L_t^\infty \dot H^{s_c}}\\
\lesssim& N_0^{-d+4+\frac{4}{p}+\frac{d}{q}},
\end{aligned}
\tag{4.11}
$$

where $1-\frac{1}{q}=\frac{1}{r}+\frac{4}{d}$.

To estimate the contribution of the second term on the right-hand side of (4.10) to (4.9), we first note that

$$
\big\|P_N\varnothing\Big(\sum_{k=0}^{p}u^k_{<\frac{N}{10p}}u^{p+1-k}_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\Big)\big\|_{L_t^\infty L_x^{q'}} \lesssim \big\|\varnothing\big(u^{p+1}_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\big)\big\|_{L_t^\infty L_x^{q'}}+\big\|\varnothing\big(u^p_{<\frac{N}{10p}}u_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\big)\big\|_{L_t^\infty L_x^{q'}}.
$$

Using Hölder's inequality, Bernstein's inequality, the assumption (4.1) and compactness (4.3), we estimate

$$
\begin{aligned}
\|\varnothing\big(u^{p+1}_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\big)\|_{L_t^\infty L_x^{q'}} \lesssim& \|u_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\|^{p-1}_{L_t^\infty L_x^{\frac{d}{4}p}}\sum_{\frac{N}{10p}\leqslant M_1\leqslant M_2\leqslant N_0}\|u_{M_1}\|_{L_t^\infty L_x^q}\|u_{M_2}\|_{L_t^\infty L_x^r}\\
\lesssim& \eta^{p-1}\sum_{\frac{N}{10p}\leqslant M_1\leqslant M_2\leqslant N_0}\|u_{M_1}\|_{L_t^\infty L_x^q}M_2^{-d+4+\frac{2d}{q}}\|u_{\leqslant N_0}\|_{L_t^\infty\dot H^{s_c}}\\
\lesssim& \eta^p N^{-d+4+\frac{4}{p}+\frac{d}{q}}\sum_{\frac{N}{10p}\leqslant M\leqslant N_0}\Big(\frac{N}{M}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}}A_q(M),
\end{aligned}
\tag{4.12}
$$

where $1-\frac{1}{q}=\frac{4(p-1)}{pd}+\frac{1}{q}+\frac{1}{r}$, and we use the fact $q>\frac{2d}{d-4}$ in the last inequality.

Similarly, we estimate

$$
\begin{aligned}
&\big\|\varnothing\big(u^p_{<\frac{N}{10p}}u_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\big)\big\|_{L_t^\infty L_x^{q'}}\\
\lesssim&\|u_{\frac{N}{10p}\leqslant\cdot\leqslant N_0}\|_{L_t^\infty L_x^2}\sum_{M_1\leqslant\cdots\leqslant M_p\leqslant\frac{N}{10p}}\prod_{j=1}^{p-1}\|u_{M_j}\|_{L^\infty_{t,x}}\|u_{M_p}\|_{L_t^\infty L_x^{\frac{2q}{q-2}}}\\
\lesssim&\eta^2N^{-s_c}\sum_{M_1\leqslant\cdots\leqslant M_p\leqslant\frac{N}{10p}}\prod_{j=1}^{p-1}M_j^{\frac{4}{p}}A_q(M_j)M_p^{-\frac{d}{2}+\frac{d}{q}+\frac{4}{p}}\\
\lesssim&\eta^2N^{-s_c}\sum_{M_1\leqslant\cdots\leqslant M_{p-1}\leqslant\frac{N}{10p}}M_1^{\varepsilon(p-1)}M_{p-1}^{-\frac{d}{2}+\frac{d}{q}+\frac{4}{p}}\\
&\times\Big(M_1^{(\frac{4}{p}-\varepsilon)(p-1)}A_q(M_1)^{p-1}+M_2^{\frac{4(p-1)}{p}}A_q(M_2)^{p-1}+\cdots+M_{p-1}^{\frac{4(p-1)}{p}}A_q(M_{p-1})^{p-1}\Big)\\
\lesssim&\eta^2N^{-d+4+\frac{4}{p}+\frac{d}{q}}\sum_{M\leqslant\frac{N}{10p}}\Big(\frac{M}{N}\Big)^{-\frac{d}{2}+4+\frac{d}{q}-}A_q(M)^{p-1}\\
\lesssim&\eta^pN^{-d+4+\frac{4}{p}+\frac{d}{q}}\sum_{M\leqslant\frac{N}{10p}}\Big(\frac{M}{N}\Big)^{-\frac{d}{2}+4+\frac{d}{q}-}A_q(M),
\end{aligned}
$$

where ε is a sufficiently small positive constant, and we use $q>\frac{2d}{d-4}$ and $A_q(M)\lesssim\eta$ with $M\leqslant N_0$ in the above inequality. This together with (4.11), (4.12) and (4.9) imply the claim (4.5). And thus, we conclude Proposition 4.1 for p being an even integer.

Case 2: p not even. Now we turn to consider that p is not an even integer and $s_c\in(2,2+p)$.

By the same argument as above, we estimate

$$
A_q(N)\lesssim N^{d-4-\frac{4}{p}-\frac{d}{q}}\|P_Nf(u)\|_{L_t^\infty L_x^{q'}},\ q>\frac{2d}{d-4}. \tag{4.13}
$$

For $N\leqslant N_0$, using the fundamental Theorem of Calculus, we decompose $f(u)$ by

$$
\begin{aligned}
f(u)=&\varnothing\big(u_{>N_0}\cdot u^p_{\leqslant N_0}\big)+\varnothing\big(u^{p+1}_{>N_0}\big)+f\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\big) &(4.14)\\
&+u_{\leqslant\frac{N}{10}}\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{<\frac{N}{10}}\big)d\theta &(4.15)\\
&+\overline{u_{\leqslant\frac{N}{10}}}\int_0^1 f_{\bar z}\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{<\frac{N}{10}}\big)d\theta. &(4.16)
\end{aligned}
$$

The contribution to the right-hand side of (4.13) coming from that contain at least one copy of $u_{>N_0}$ can be estimated by the same argument as (4.11).

By a simple computation, we have the following equivalence for p being not an even integer

$$s_c < 2+p \iff 2p^2-(d-4)p+8>0,$$

$$\begin{cases} \frac{8}{d-4} < p \leqslant 1, \\ 2 < s_c < 2+p \end{cases} \iff \begin{cases} d=13,\ p\leqslant 1 < p_1 := \frac{(d-4)-\sqrt{(d-4)^2-64}}{4} \\ d\geqslant 14,\ p<p_1 \end{cases}$$

$$\begin{cases} p > \max\{1, \frac{8}{d-4}\}, \\ 2 < s_c < 2+p \end{cases} \iff \begin{cases} 9\leqslant d\leqslant 12:\ p>\frac{8}{d-4}, \\ d=13:\ 1<p<p_1 \text{ or } p>p_2, \\ d\geqslant 14:\ p>p_2 := \frac{(d-4)+\sqrt{(d-4)^2-64}}{4}. \end{cases}$$

Next, we divide two cases to estimate the contribution coming from the remain terms.
Subcase 2(i): $p\leqslant 1$. In this case we have only $f_z(u)\in C^p(\mathbb{C})$.

We first consider the contribution coming from the term $f\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}\big)$. Using $l^p\subset l^1$, Hölder's inequality, Bernstein's inequality and compactness (4.3), we deduce that

$$\begin{aligned}
&\|f(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0})\|_{L_t^\infty L_x^{q'}} \\
\lesssim & \sum_{\frac{N}{10}\leqslant M_1\leqslant N_0} \big\|u_{M_1}|u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}|^p\big\|_{L_t^\infty L_x^{q'}} \\
\lesssim & \sum_{\frac{N}{10}\leqslant M_1,M_2\leqslant N_0} \big\|u_{M_1}|u_{M_2}|^p\big\|_{L_t^\infty L_x^{q'}} \\
\lesssim & \sum_{\frac{N}{10}\leqslant M_1\leqslant M_2\leqslant N_0} \|u_{M_1}\|_{L_t^\infty L_x^q}\|u_{M_2}\|^p_{L_t^\infty L_x^{\frac{pq}{q-2}}} \\
&+ \sum_{\frac{N}{10}\leqslant M_2\leqslant M_1\leqslant N_0} \|u_{M_1}\|^p_{L_t^\infty L_x^{\frac{pq}{q-2}}}\|u_{M_1}\|^{1-p}_{L_t^\infty L_x^q}\|u_{M_2}\|^p_{L_t^\infty L_x^q} \\
\lesssim & \eta^p \sum_{\frac{N}{10}\leqslant M\leqslant N_0} M^{-d+4+\frac{4}{p}+\frac{d}{q}} A_q(M) \\
&+\eta^p \sum_{\frac{N}{10}\leqslant M_2\leqslant M_1\leqslant N_0} \Big(\frac{M_2}{M_1}\Big)^{2p(d-4-\frac{4}{p}-\frac{d}{q})} \big(M_1^{-d+4+\frac{4}{p}+\frac{d}{q}} A_q(M_1)\big)^{1-p}\big(M_2^{-d+4+\frac{4}{p}+\frac{d}{q}} A_q(M_2)\big)^p \\
\lesssim & \eta^p N^{-d+4+\frac{4}{p}+\frac{d}{q}} \sum_{\frac{N}{10}\leqslant M\leqslant N_0} \Big(\frac{N}{M}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}} A_q(M).
\end{aligned}$$

Now we consider the contribution coming from (4.15) and (4.16). It suffices to consider (4.15), since similar arguments can be used to deal with (4.16). By Hölder's

inequality, we obtain

$$\begin{aligned}
&\Big\|P_N\Big(u_{\leqslant \frac{N}{10}}\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{<\frac{N}{10}}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{q'}}\\
\lesssim&\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^r}\Big\|P_N\Big(\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{<\frac{N}{10}}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{\frac{d}{p+4}}}\\
\lesssim&\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^r}\big\|P_N\big(f_z(u_{\leqslant N_0})\big)\big\|_{L_t^\infty L_x^{\frac{d}{p+4}}},
\end{aligned}\tag{4.17}$$

where $1-\frac{1}{q}=\frac{1}{r}+\frac{p+4}{d}$. On the other hand, it follows from the nonlinear Bernstein inequality (2.16) that

$$\big\|P_N\big(f_z(u_{\leqslant N_0})\big)\big\|_{L_t^\infty L_x^{\frac{d}{p+4}}}\lesssim N^{-p}\big\|\nabla u_{\leqslant N_0}\big\|^p_{L_t^\infty L_x^{\frac{pd}{p+4}}}\lesssim N^{-p}\big\||\nabla|^{s_c}u_{\leqslant N_0}\big\|^p_{L_t^\infty L_x^2}.$$

Plugging this into (4.17), and by Bernstein's inequality, compactness (4.3) we derive

$$\begin{aligned}
&\Big\|P_N\Big(u_{\leqslant \frac{N}{10}}\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{\leqslant N_0}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{q'}}\\
\lesssim&N^{-p}\big\||\nabla|^{s_c}u_{\leqslant N_0}\big\|^p_{L_t^\infty L_x^2}\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^r}\\
\lesssim&\eta^pN^{-d+4+\frac{4}{p}+\frac{d}{q}}\sum_{M<\frac{N}{10}}\Big(\frac{M}{N}\Big)^{p+4+\frac{d}{q}+\frac{4}{p}-d}A_q(M).
\end{aligned}$$

Putting everything together, we deduce that $A_q(N)$ satisfies the following recurrence formula

$$\begin{aligned}
A_q(N)\lesssim&\Big(\frac{N}{N_0}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}}+\eta^p\sum_{\frac{N}{10}\leqslant M\leqslant N_0}\Big(\frac{N}{M}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}}A_q(M)\\
&+\eta^p\sum_{M\leqslant\frac{N}{10}}\Big(\frac{M}{N}\Big)^{-d+4+p+\frac{d}{q}+\frac{4}{p}}A_q(M)
\end{aligned}\tag{4.18}$$

for any $q\in\big(\frac{2d}{d-4},\frac{d}{d-4-p-\frac{4}{p}}\big)$. Applying Lemma 4.1 again, we obtain

$$A_q(N)\lesssim N^{(d-4-\frac{4}{p}-\frac{d}{q})-},\ q\in\Big(\frac{2d}{d-4},\frac{d}{d-4-p-\frac{4}{p}}\Big)\tag{4.19}$$

which means for $N\leqslant N_0$

$$\|u_N\|_{L_x^q}\lesssim N^{\left(d-4-\frac{2d}{q}\right)-},\ q\in\Big(\frac{2d}{d-4},\frac{d}{d-4-p-\frac{4}{p}}\Big).\tag{4.20}$$

This together with Remark 4.1 (ii) yields that

$$u\in L_t^\infty(\mathbb{R};L_x^q(\mathbb{R}^d)),\ q\in\Big(\frac{2d}{d-4},\min\Big\{\frac{d}{d-4-p-\frac{4}{p}},\frac{dp}{4}\Big\}\Big).$$

Interpolating this with $u\in L_t^\infty L_x^{\frac{pd}{4}}(\mathbb{R}\times\mathbb{R}^d)$, we conclude Proposition 4.1 for $p\in\big(\frac{8}{d-4},1\big]$.

Subcase 2(ii): $p > \max\{1, \frac{8}{d-4}\}$**.** By the same argument as (4.12), we estimate

$$\|f(u_{\frac{N}{10p}\leqslant\cdot\leqslant N_0})\|_{L_t^\infty L_x^{q'}} \lesssim \eta^p N^{-d+4+\frac{4}{p}+\frac{d}{q}} \sum_{\frac{N}{10}\leqslant M\leqslant N_0} \Big(\frac{N}{M}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}} A_q(M).$$

Next, we consider the contribution coming from (4.15) and (4.16). It suffices to consider (4.15), since similar arguments can be used to deal with (4.16). Given p, there exists $\varepsilon > 0$ such that $s_c < 2 + p - \varepsilon$. Using the Hölder, Bernstein's inequalities and compactness (4.3), we derive that

$$\begin{aligned}
&\Big\|P_N\Big(u_{\leqslant\frac{N}{10}}\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{<\frac{N}{10}}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{q'}} \qquad (4.21)\\
\lesssim&\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^{r_1}}\Big\|P_{>\frac{N}{10}}\Big(\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0}+\theta u_{<\frac{N}{10}}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{r_2}}\\
\lesssim&N^{-s_c+2-\varepsilon}\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^{r_1}}\big\||\nabla|^{s_c-2+\varepsilon}u_{\leqslant N_0}\big\|_{L_t^\infty L_x^{\frac{2d}{d-4+2\varepsilon}}}\|u_{\leqslant N_0}\|^{p-1}_{L_t^\infty L_x^{\frac{pd}{4}}}\\
\lesssim&\eta^p N^{-d+4+\frac{d}{q}+\frac{4}{p}}\sum_{M\leqslant\frac{N}{10}}\Big(\frac{M}{N}\Big)^{-\frac{d}{2}+2+\varepsilon+\frac{d}{q}}A_q(M),
\end{aligned}$$

where $1-\frac{1}{q}=\frac{1}{r_1}+\frac{1}{r_2}$ and $\frac{1}{r_2}=\frac{2d}{d-4+\varepsilon}+\frac{4(p-1)}{pd}$. Thus, we derive that $A_q(N)$ satisfies the following recurrence formula

$$\begin{aligned}
A_q(N)\lesssim&\Big(\frac{N}{N_0}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}}+\eta^p\sum_{\frac{N}{10}\leqslant M\leqslant N_0}\Big(\frac{N}{M}\Big)^{d-4-\frac{4}{p}-\frac{d}{q}}A_q(M)\\
&+\eta^p\sum_{M\leqslant\frac{N}{10}}\Big(\frac{M}{N}\Big)^{-\frac{d}{2}+2+\varepsilon+\frac{d}{q}}A_q(M)
\end{aligned} \tag{4.22}$$

for any $q\in\big(\frac{2d}{d-4},\frac{2d}{d-4-2\varepsilon}\big)$. Applying Lemma 4.1 again, we get

$$A_q(N)\lesssim N^{(d-4-\frac{4}{p}-\frac{d}{q})-},\ q\in\Big(\frac{2d}{d-4},\frac{2d}{d-4-2\varepsilon}\Big) \tag{4.23}$$

which means for $N\leqslant N_0$

$$\|u_N\|_{L_x^q}\lesssim N^{\left(d-4-\frac{2d}{q}\right)-},\ q\in\Big(\frac{2d}{d-4},\frac{2d}{d-4-2\varepsilon}\Big). \tag{4.24}$$

This together with Remark 4.1 (ii) yields that

$$u\in L_t^\infty(\mathbb{R};L_x^q(\mathbb{R}^d)),\ q\in\Big(\frac{d}{d-4-\frac{4}{p}},\min\Big\{\frac{2d}{d-4-2\varepsilon},\frac{dp}{4}\Big\}\Big).$$

Interpolating this with $u\in L_t^\infty L_x^{\frac{pd}{4}}(\mathbb{R}\times\mathbb{R}^d)$, we conclude Proposition 4.1 for $p>\max\{1,\frac{8}{d-4}\}$.

Therefore, we complete the proof of Proposition 4.1.

The next result shows the additional decay for the energy-subcritical and energy-critical cases.

Proposition 4.2 (Additional decay II)**.** *Let $d \geqslant 9$ and $1 \leqslant s_c \leqslant 2$. And let u be a global solution to* (1.1) *that is almost periodic modulo symmetries. Assume also $\inf_{t\in\mathbb{R}} N(t) \geqslant 1$. Then we have*

(4.25)
$$u \in L_t^\infty(\mathbb{R}; L_x^q(\mathbb{R}^d)), \quad q \in \begin{cases} \Big(r_1, \frac{pd}{4}\Big], \ r_1 = \frac{2d(\frac{8}{p}-d+6+s_c)}{d(\frac{8}{p}-d+6)+2s_c(d-5-\frac{4}{p})}, & \textit{if } p > 1, \\ \Big(r_2, \frac{pd}{4}\Big], \ r_2 = \frac{2d(2p+\frac{8}{p}-d+4+s_c)}{d(2p+\frac{8}{p}-d+4)+2s_c(d-4-\frac{4}{p}-p)}, & \textit{if } p \leqslant 1. \end{cases}$$

In particular, if $s_c = 2$; that is: $p = \frac{8}{d-4}$, then

$$u \in L_t^\infty(\mathbb{R}; L_x^q(\mathbb{R}^d)), \quad q \in \begin{cases} \Big(\frac{2d}{d-3}, \frac{2d}{d-4}\Big], & \textit{if } p > 1, \ \textit{i.e. } d < 12, \\ \Big(\frac{2(d+4)}{d}, \frac{2d}{d-4}\Big], & \textit{if } p \leqslant 1, \ \textit{i.e. } d \geqslant 12. \end{cases} \tag{4.26}$$

Remark 4.2. *It is easy to check that*

$$r_1, r_2 < \frac{pd}{4}, \ \textit{whenever } p > \frac{8}{d}.$$

The proof of Proposition 4.2: Noting that $\frac{2d}{d-4} \leqslant \frac{d}{d-4-\frac{4}{p}}$ in this case and by the similar argument as Proposition 4.1, we have for $N \leqslant N_0$

$$\|u_N\|_{L_t^\infty L_x^q} \lesssim N^{\left(d-4-\frac{2d}{q}\right)-}, \ q \in \begin{cases} \Big(\frac{d}{d-4-\frac{4}{p}}, \frac{d}{d-5-\frac{4}{p}}\Big), & \text{if } p > 1, \\ \Big(\frac{d}{d-4-\frac{4}{p}}, \frac{d}{d-4-p-\frac{4}{p}}\Big), & \text{if } p \leqslant 1. \end{cases} \tag{4.27}$$

We remark that one estimates the term (4.21) in the case $p > 1$ by a different way. Term (4.21) is estimated by

$$\begin{aligned}
&\Big\|P_N\Big(u_{\leqslant \frac{N}{10}} \int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0} + \theta u_{<\frac{N}{10}}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{q'}} \\
\lesssim & \|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^r} \Big\|P_{>\frac{N}{10}}\Big(\int_0^1 f_z\big(u_{\frac{N}{10}\leqslant\cdot\leqslant N_0} + \theta u_{<\frac{N}{10}}\big)d\theta\Big)\Big\|_{L_t^\infty L_x^{\frac{d}{5}}} \\
\lesssim & N^{-1}\|u_{<\frac{N}{10}}\|_{L_t^\infty L_x^r} \big\|\nabla u_{\leqslant N_0}\big\|_{L_t^\infty L_x^{\frac{pd}{p+4}}} \|u_{\leqslant N_0}\|^{p-1}_{L_t^\infty L_x^{\frac{pd}{4}}} \\
\lesssim & \eta^p N^{-d+4+\frac{d}{q}+\frac{4}{p}} \sum_{M\leqslant\frac{N}{10}} \Big(\frac{M}{N}\Big)^{-d+5+\frac{d}{q}+\frac{4}{p}} A_q(M).
\end{aligned}$$

Therefore, we get (4.27).

Case 1: $p > 1$**.** We have by (4.27) with $q = \frac{d}{d-5-\frac{4}{p}}-$

$$\|u_N\|_{L_t^\infty L_x^{\frac{d}{d-5-\frac{4}{p}}-}} \lesssim N^{(\frac{8}{p}-d+6)-}. \tag{4.28}$$

On the other hand, from the assumption (1.10): $u \in L_t^\infty \dot{H}_x^{s_c}(\mathbb{R}\times\mathbb{R}^d)$, we know that

$$\|u_N\|_{L_t^\infty L_x^2} \lesssim N^{-s_c}.$$

Interpolating this with (4.28), we deduce that for $N \leqslant N_0$

$$\|u_N\|_{L_t^\infty L_x^{r_1+}} \lesssim N^{0+}.$$

Hence we obtain $u \in L_t^\infty L_x^{r_1+}(\mathbb{R}\times\mathbb{R}^d)$ by Remark 4.1 (ii). Interpolating this with $u \in L_t^\infty \dot H^{s_c} \subset L_t^\infty L_x^{\frac{pd}{4}}$ again, we derive that

$$u \in L_t^\infty(\mathbb{R}; L_x^q(\mathbb{R}^d)),\ q \in \Big(r_1, \frac{pd}{4}\Big],\ r_1 = \frac{2d(\frac{8}{p} - d + 6 + s_c)}{d(\frac{8}{p} - d + 6) + 2s_c(d - 5 - \frac{4}{p})}.$$

Case 2: $p \leqslant 1$**.** By (4.27), we get

$$\|u_N\|_{L_t^\infty L_x^{\frac{d}{d-4-p-\frac{4}{p}}-}} \lesssim N^{(2p+\frac{8}{p}-d+4)}. \tag{4.29}$$

On the other hand, from the assumption (1.10): $u \in L_t^\infty \dot H_x^{s_c}(\mathbb{R}\times\mathbb{R}^d)$, we know that

$$\|u_N\|_{L_t^\infty L_x^2} \lesssim N^{-s_c}.$$

Combining this with (4.29), we obtain for $N \leqslant N_0$

$$\|u_N\|_{L_t^\infty L_x^{r_2+}} \lesssim N^{0+}.$$

This implies $u \in L_t^\infty L_x^{r_2+}$ by Remark 4.1 (ii). Interpolating this with $u \in L_t^\infty L_x^{\frac{pd}{4}}$ concludes the proof of this proposition.

4.2. **Negative regularity.** Now we utilize the double Duhamel trick to show Theorem 1.3. First, we drive a preliminary lemma.

Lemma 4.2. *Suppose that $u \in L_t^\infty(\mathbb{R}; \dot H_x^s(\mathbb{R}^d))$ for some $s \in [0, s_c]$. Assume also that there exists a positive constant α independent of s such that*

$$\big\||\nabla|^s P_N u\big\|_{L_t^\infty L_x^2} \lesssim_s N^\alpha,\ \forall\ N \leqslant 1. \tag{4.30}$$

Then, for any $\beta \in [0, \alpha)$, we have $u \in L_t^\infty(\mathbb{R}; \dot H_x^{s-\beta}(\mathbb{R}^d))$.

Proof. Using Bernstein's inequality and the assumption $u \in L_t^\infty(\mathbb{R}; \dot H_x^s(\mathbb{R}^d))$, we have

$$\begin{aligned}\big\||\nabla|^{s-\beta} u\big\|_{L_t^\infty L_x^2} &\lesssim \sum_{N\leqslant 1} \big\||\nabla|^{s-\beta} P_N u\big\|_{L_t^\infty L_x^2} + \big\||\nabla|^{s-\beta} P_{\geqslant 1} u\big\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{N\leqslant 1} N^{-\beta}\big\||\nabla|^s P_N u\big\|_{L_t^\infty L_x^2} + \big\||\nabla|^s u\big\|_{L_t^\infty L_x^2}\\ &\lesssim \sum_{N\leqslant 1} N^{\alpha-\beta} + 1 < +\infty.\end{aligned}$$

This completes the proof of Lemma 4.2. □

The proof of Theorem 1.3: From Lemma 4.2, we know that the proof of Theorem 1.3 is reduced to show that for any $s \in [0, s_c]$, there exists a positive constant α independent of s such that

$$\big\||\nabla|^s u_N\big\|_{L_t^\infty(\mathbb{R};L_x^2)} \lesssim N^\alpha. \tag{4.31}$$

Indeed, we first apply (4.31) with $s = s_c$. Then we conclude that $u \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c-\alpha+})$ by Lemma 4.2. And then we apply (4.31) with $s = s_c - \alpha+$ and obtain $u \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c-2\alpha+})$. Iterating this procedure finitely many times, we derive $u \in L_t^\infty(\mathbb{R}; \dot{H}_x^{-\varepsilon})$ for any $0 < \varepsilon < \alpha$.

Hence it remains to prove the claim (4.31). We divide two cases to discuss. First, we consider the energy-supercritical case.

Case 1: $s_c > 2$ (energy-supercritical). Assume that $u \in L_t^\infty(\mathbb{R}; \dot{H}_x^s(\mathbb{R}^d))$ for some $0 \leqslant s \leqslant s_c$. It follows from the additional decay (Proposition 4.1) that

$$u \in L_t^\infty(\mathbb{R}; L_x^q(\mathbb{R}^d)), \quad q \in \Big(\frac{2d}{d-4}, \frac{d}{4}p\Big]. \tag{4.32}$$

And so, we obtain by (2.13)

$$\big\||\nabla|^s f(u)\big\|_{L_t^\infty L_x^r} \lesssim \big\||\nabla|^s u\big\|_{L_t^\infty L_x^2} \|u\|_{L_t^\infty L_x^q}^p \lesssim 1, \quad r = \frac{2q}{q+2p} < 2. \tag{4.33}$$

Then the condition $r \geqslant 1$ requires $q \geqslant 2p$.

It follows from No waste Duhamel formula (1.15) that

$$u(0) = -\int_0^\infty e^{it\Delta^2} f(u)(t)dt = \int_{-\infty}^0 e^{i\tau\Delta^2} f(u)(\tau)d\tau.$$

And so

$$\begin{aligned}
\big\||\nabla|^s u_N(0)\big\|_2^2 &= -\Big\langle \int_0^\infty e^{it\Delta^2}|\nabla|^s P_N f(u)(t)dt, \int_{-\infty}^0 e^{i\tau\Delta^2}|\nabla|^s P_N f(u)(\tau)d\tau\Big\rangle \\
&= -\int_0^\infty \int_{-\infty}^0 \Big\langle e^{i(t-\tau)\Delta^2}|\nabla|^s P_N f(u)(t), |\nabla|^s P_N f(u)(\tau)\Big\rangle d\tau dt \\
&\triangleq \int_0^\infty \int_{-\infty}^0 F(t,\tau) d\tau dt,
\end{aligned} \tag{4.34}$$

where

$$F(t,\tau) = -\Big\langle e^{i(t-\tau)\Delta^2}|\nabla|^s P_N f(u)(t), |\nabla|^s P_N f(u)(\tau)\Big\rangle.$$

On one hand, using the Hölder, Bernstein inequalities and (4.33), we get

$$\begin{aligned}
F(t,\tau) \leqslant& \big\|e^{i(t-\tau)\Delta^2}|\nabla|^s P_N f(u)\big\|_{L_x^2} \big\||\nabla|^s P_N f(u)\big\|_{L_x^2} \\
\lesssim& N^{2d\left(\frac{1}{r}-\frac{1}{2}\right)} \big\||\nabla|^s P_N f(u)\big\|_{L_x^r}^2 \\
\lesssim& N^{2d\left(\frac{1}{r}-\frac{1}{2}\right)}.
\end{aligned} \tag{4.35}$$

On the other hand, by Hölder's inequality and dispersive estimate (2.3), we derive

$$\begin{aligned}F(t,\tau)\leqslant&\big\|e^{i(t-\tau)\Delta^2}|\nabla|^sP_Nf(u)\big\|_{L_x^{r'}}\big\||\nabla|^sP_Nf(u)\big\|_{L_x^r}\\ \lesssim&(t-\tau)^{-d\left(\frac14-\frac{1}{2r'}\right)}\big\||\nabla|^sP_Nf(u)\big\|_{L_x^r}^2\\ \lesssim&(t-\tau)^{-d\left(\frac{1}{2r}-\frac14\right)}.\end{aligned}\tag{4.36}$$

Hence, plugging (4.35) and (4.36) into (4.34), we obtain

$$\begin{aligned}\big\||\nabla|^su_N(0)\big\|_2^2\lesssim&\int_0^\infty\int_{-\infty}^0\min\Big\{N^{2d\left(\frac1r-\frac12\right)},(t-\tau)^{-d\left(\frac{1}{2r}-\frac14\right)}\Big\}d\tau dt\\ \lesssim&\frac14\int_{\mathbb{R}}\int_{\mathbb{R}}\min\Big\{N^{2d\left(\frac1r-\frac12\right)},(|t|+|\tau|)^{-d\left(\frac{1}{2r}-\frac14\right)}\Big\}d\tau dt\\ \lesssim&N^{\frac{2pd}{q}}\iint_{|t|+|\tau|\leqslant N^{-4}}d\tau dt+\iint_{|t|+|\tau|\geqslant N^{-4}}(|t|+|\tau|)^{-\frac{pd}{2q}}dtd\tau\\ \lesssim&N^{-8+\frac{2pd}{q}},\end{aligned}\tag{4.37}$$

where $r=\frac{2q}{q+2p}$ and we also need the restriction $\frac{pd}{2q}>2$ to guarantee the above integral converges. Therefore,

$$\big\||\nabla|^su_N(0)\big\|_2\lesssim N^{-4+\frac{pd}{q}},\quad q\in\Big[2p,\frac{pd}{4}\Big)\bigcap\Big(\frac{2d}{d-4},\frac{pd}{4}\Big].\tag{4.38}$$

The condition $2p<\frac{pd}{4}$ requires the dimension d such that $d\geqslant 9$. Now if we take $q=\max\big\{2p,\frac{2d}{d-4}+\big\}$, then we obtain

$$\alpha=-4+\frac{pd}{q}=\min\Big\{-4+\frac d2,-4+\frac{(d-4)p}{2}-\Big\}>0.$$

Therefore we conclude (4.31). And so we complete the proof of Theorem 1.3 for $s_c>2$.

We remark that the balance between the bounds provided by Lemma 4.1 and the bound required by Theorem 1.3 is the source of our restriction to dimensions $d\geqslant 9$. As we noted above, (4.33) provides the $L_t^\infty L_x^q$ bounds for $q\geqslant 2p$, while (4.37) requires this bound with $q<\frac{pd}{4}$. These conditions on q impose the restriction $d\geqslant 9$.

Case 2: $1\leqslant s_c\leqslant 2$ **(energy-subcritical and energy-critical).** By the same argument as the energy-supercritical case, we have

$$\big\||\nabla|^su_N(0)\big\|_2\lesssim N^{-4+\frac{pd}{q}},$$

where q satisfies

$$q\in\Big[2p,\frac{pd}{4}\Big)\bigcap\begin{cases}\Big(r_1,\frac{pd}{4}\Big], & \text{if } p>1,\\ \Big(r_2,\frac{pd}{4}\Big], & \text{if } p\leqslant 1.\end{cases}\tag{4.39}$$

where (r_1,r_2) is as in Proposition 4.2. If we take $q=\frac{pd}{4}-$, then we obtain $\alpha=-4+\frac{pd}{q}=0+>0$. Hence we get (4.31). Therefore, we complete the proof of Theorem 1.3 for $s_c\in[1,2]$. Therefore, we conclude Theorem 1.1.

5. Appendix

In this appendix, we show the perturbation theory. We first consider that p is an even integer.

5.1. **Perturbation I: p even.**

Here we give a perturbation result under the weakest assumption on the difference of the initial data (5.3), since it is easy to show the smallness assumption (5.3) can be derived from (2.24) by Strichartz estimate.

Lemma 5.1 (Perturbation Lemma, p even)**.** *Assume that $s_c = \frac{d}{2} - \frac{4}{p} \geqslant 1$, and p is an even integer. Let I be a compact time interval and u, $\tilde{u}$ satisfy*

$$(i\partial_t + \Delta^2)u = -f(u) + eq(u)$$
$$(i\partial_t + \Delta^2)\tilde{u} = -f(\tilde{u}) + eq(\tilde{u})$$

for some function $eq(u), eq(\tilde{u})$, and $f(u) = |u|^p u$. Assume that for some constants $M, E > 0$, we have

$$\|u\|_{L_t^\infty(I;\dot{H}_x^{s_c}(\mathbb{R}^d))} + \|\tilde{u}\|_{L_t^\infty(I;\dot{H}_x^{s_c}(\mathbb{R}^d))} \leqslant E, \tag{5.1}$$

$$S_I(\tilde{u}) \leqslant M, \tag{5.2}$$

Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\big\||\nabla|^{s_c-1} e^{i(t-t_0)\Delta^2}(u-\tilde{u})(t_0)\big\|_{L_t^{2(p+1)} L_x^{\frac{2d(p+1)}{(d-2)(p+1)-4}}(I\times\mathbb{R}^d)} \leqslant \varepsilon, \tag{5.3}$$

where $0 < \varepsilon < \varepsilon_1 = \varepsilon_1(M, E)$ is a small constant. Assume also that we have smallness conditions

$$\big\||\nabla|^{s_c-1}\big(eq(u), eq(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \leqslant \varepsilon, \tag{5.4}$$

where ε is as above.

Then we conclude that

$$\begin{aligned} S_I(u-\tilde{u}) \leqslant& C(M,E)\varepsilon \\ \big\||\nabla|^{s_c}(u-\tilde{u})\big\|_{S^0(I)} \leqslant& C(M,E)\varepsilon \\ \big\||\nabla|^{s_c}u\big\|_{S^0(I)} \leqslant& C(M,E). \end{aligned} \tag{5.5}$$

Proof. Since $S_I(\tilde{u}) \leqslant M$, we may subdivide I into $C(M, \varepsilon_0)$ time intervals I_j such that

$$S_{I_j}(\tilde{u}) \leqslant \epsilon_0 \ll 1, \qquad 1 \leqslant j \leqslant C(M, \varepsilon_0).$$

By the Strichartz estimate and standard bootstrap argument we have

$$\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I_j)} \leqslant C(E), \qquad 1 \leqslant j \leqslant C(M, \varepsilon_0).$$

Summing up over all the intervals, we obtain that

$$\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I)} \leqslant C(E, M). \tag{5.6}$$

In particular, we have by Sobolev embedding

$$\|\tilde{u}\|_{Z(I)} := \big\||\nabla|^{s_c-1}\tilde{u}\big\|_{L_t^{2(p+1)} L_x^{\frac{2d(p+1)}{(d-2)(p+1)-4}}(I\times\mathbb{R}^d)} \leqslant C(E, M), \tag{5.7}$$

which implies that there exists a partition of the right half of I at t_0:

$$t_0 < t_1 < \cdots < t_N,\ I_j = (t_j, t_{j+1}),\ I \cap (t_0, \infty) = (t_0, t_N),$$

such that $N \leqslant C(L, \delta)$ and for any $j = 0, 1, \cdots, N-1$, we have

$$\|\tilde{u}\|_{Z(I_j)} \leqslant \delta \ll 1. \tag{5.8}$$

The estimate on the left half of I at t_0 is analogue, we omit it.

Let

$$\gamma(t) = u(t) - \tilde{u}(t), \tag{5.9}$$

and

$$\gamma_j(t) = e^{i(t-t_j)\Delta^2}\gamma(t_j),\ 0 \leqslant j \leqslant N-1, \tag{5.10}$$

then γ satisfies the following difference equation

$$\begin{cases} (i\partial_t + \Delta^2)\gamma = -f(\tilde{u} + \gamma) + f(\tilde{u}) + eq(u) - eq(\tilde{u}), \\ \gamma(t_j) = \gamma_j(t_j), \end{cases}$$

which implies that

$$\begin{aligned} \gamma(t) =& \gamma_j(t) - i \int_{t_j}^{t} e^{i(t-s)\Delta^2}\big(- f(\tilde{u} + \gamma) + f(\tilde{u}) + eq(u) - eq(\tilde{u})\big) ds, \\ \gamma_{j+1}(t) =& \gamma_j(t) - i \int_{t_j}^{t_{j+1}} e^{i(t-s)\Delta^2}\big(- f(\tilde{u} + \gamma) + f(\tilde{u}) + eq(u) - eq(\tilde{u})\big) ds. \end{aligned}$$

It follows from Strichartz estimate and nonlinear estimate (2.13) that

(5.11)

$$\begin{aligned} &\|\gamma - \gamma_j\|_{Z(I_j)} + \|\gamma_{j+1} - \gamma_j\|_{Z(I)} \\ \lesssim& \big\||\nabla|^{s_c-1}\big(f(\tilde{u} - \gamma) + f(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} + \big\||\nabla|^{s_c-1}\big(eq(u), eq(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \\ \lesssim& \sum_{k=1}^{p+1} \|\gamma\|_{Z(I_j)}^k \|\tilde{u}\|_{Z(I_j)}^{p+1-k} + \big\||\nabla|^{s_c-1}\big(eq(u), eq(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \\ \lesssim& \sum_{k=1}^{p+1} \|\gamma\|_{Z(I_j)}^k \|\tilde{u}\|_{Z(I_j)}^{p+1-k} + \varepsilon. \end{aligned}$$

Therefore, assuming that

$$\|\gamma\|_{Z(I_j)} \leqslant \delta \ll 1,\ \forall\ j = 0, 1, \cdots, N-1, \tag{5.12}$$

then by (5.8) and (5.11), we have

$$\|\gamma\|_{Z(I_j)} + \|\gamma_{j+1}\|_{Z(t_{j+1}, t_N)} \leqslant C\|\gamma_j\|_{Z(t_j, t_N)} + \varepsilon, \tag{5.13}$$

for some absolute constant $C > 0$. By (5.3) and iteration on j, we obtain

$$\|\gamma\|_{Z(I)} \leqslant (2C)^N \varepsilon \leqslant \frac{\delta}{2}, \tag{5.14}$$

if we choose ε_1 sufficiently small. Hence the assumption (5.12) is justified by continuity in t and induction on j. Then repeating the estimate (5.11) once again, we can get

the critical-norm estimate on γ, which implies the Strichartz estimates on u. This concludes the proof of this lemma. $\square$

5.2. **Perturbation II: p not even.** In this subsection, we will establish the perturbation theory of the solution of (1.1) with p being not an even integer. We restate the perturbation lemma as follows.

Lemma 5.2 (Perturbation Lemma, p not even)**.** *Assume that p is not an even integer and $1 \leqslant s_c < 2 + p$. Let I be a compact time interval and u, $\tilde{u}$ satisfy*

$$\begin{aligned}(i\partial_t + \Delta^2)u &= - f(u) + eq(u)\\ (i\partial_t + \Delta^2)\tilde{u} &= - f(\tilde{u}) + eq(\tilde{u})\end{aligned}$$

for some function $eq(u), eq(\tilde{u})$, and $f(u) = |u|^p u$. Assume that for some constants $M, E > 0$, we have

$$\|u\|_{L_t^\infty(I;\dot{H}_x^{s_c}(\mathbb{R}^d))} + \|\tilde{u}\|_{L_t^\infty(I;\dot{H}_x^{s_c}(\mathbb{R}^d))} \leqslant E, \tag{5.15}$$

$$S_I(\tilde{u}) \leqslant M, \tag{5.16}$$

Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}$ in the sense that

$$\big\|u(t_0) - \tilde{u}(t_0)\big\|_{\dot{H}^{s_c}} \leqslant \varepsilon, \tag{5.17}$$

where $0 < \varepsilon < \varepsilon_1(M,E)$ is a small constant. Assume also that we have smallness conditions

$$\Big\||\nabla|^{s_c-1}\big(eq(u), eq(\tilde{u})\big)\Big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \leqslant \varepsilon, \tag{5.18}$$

where ε is as above.

Then we conclude that

$$\begin{aligned}S_I(u-\tilde{u}) &\leqslant C(M,E)\varepsilon^{c_1}\\ \big\||\nabla|^{s_c}(u-\tilde{u})\big\|_{S^0(I)} &\leqslant C(M,E)\varepsilon^{c_2}\\ \big\||\nabla|^{s_c}u\big\|_{S^0(I)} &\leqslant C(M,E),\end{aligned} \tag{5.19}$$

where c_1, c_2 are positive constants that depend on d, p, E and M.

The proof of the above lemma with $p > s_c - 1$ is similar to Lemma 5.1 based on the use of the standard Strichartz estimates. However this proof can not be applied directly to $p \leqslant s_c - 1$. The main reason for this is that for $p \leqslant s_c - 1$ the derivative of the nonlinearity is no longer Lipschitz continuous in the standard Strichartz space. In [36], Tao and Visan first overcame this problem in the context of the energy-critical NLS in dimensions $d > 6$ by making use of certain "exotic Strichartz" spaces which have same scaling with standard Strichartz space but lower derivative. Later, Killip and Visan simplified the proof in [23] where stability is established in Sobolev Strichartz spaces where they utilized the fractional chain rule.

Therefore, we always assume that $p \leqslant s_c - 1$. We give a sketch proof by the similar argument as in [19]. First, it is useful to define several spaces and give estimates of the

nonlinearities in terms of these spaces. Given $s := \frac{p}{2}$, define

$$\begin{aligned}
&\|u\|_{X^0(I)} := \|u\|_{L_t^{q_0} L_x^r(I\times\mathbb{R}^d)} \triangleq \|u\|_{L_t^{q_0} L_x^{\frac{r_0 d}{d-r_0 s_c}}(I\times\mathbb{R}^d)} = \|u\|_{L_t^{q_0} L_x^{\frac{(p+2)d}{d-p}}(I\times\mathbb{R}^d)} \\
&\|u\|_{X(I)} := \big\||\nabla|^s u\big\|_{L_t^{q_0} L_x^{r_1}(I\times\mathbb{R}^d)}, \quad r_1 = \frac{2r_0 d}{2d - r_0(2s_c - p)} \\
&\|F\|_{Y(I)} := \big\||\nabla|^s F\big\|_{L_t^{\frac{q_0}{1+p}} L_x^{r_1'}(I\times\mathbb{R}^d)}, \quad \frac{1}{r_1} + \frac{1}{r_1'} = 1,
\end{aligned} \tag{5.20}$$

where $(q_0, r_0) = \Big(\frac{4p(p+2)}{p^2 - p(d-4) + 8}, \frac{d(p+2)}{d - p + s_c(p+2)}\Big)$, $2 < r_0 < \frac{d}{s_c}$, $\frac{d+4}{4}p < q_0 < +\infty$. It is easy to check that (q_0, r, r_1, s) satisfies

(1) (q_0, r): s_c-admissible pair, that is $\frac{4}{q_0} = d\Big(\frac{1}{2} - \frac{1}{r}\Big) - s_c = \frac{4}{p} - \frac{d}{r}$.

(2) (q_0, r_1): $(s_c - s)$-admissible pair, that is $\frac{4}{q_0} = d\Big(\frac{1}{2} - \frac{1}{r_1}\Big) - (s_c - s)$.

(3) Nonlinear estimate

$$\|f(u)\|_{Y(I)} \lesssim \|u\|_{X(I)} \|u\|_{X^0(I)}^p \lesssim \|u\|_{X(I)}^{p+1} \tag{5.21}$$

requires $\frac{1}{r_1'} = \frac{1}{r_1} + \frac{p}{r}$.

(4) "Exotic Strichartz estimate" Hardy-Littlewood-Sobolev requires:

$$1 + \frac{1}{q_0} = d\Big(\frac{1}{4} - \frac{1}{2r_1}\Big) + \frac{p+1}{q_0}.$$

It is easy to verify that the Sobolev embedding relations

$$\|u\|_{X^0(I)} \lesssim \|u\|_{X(I)} \lesssim \big\||\nabla|^{s_c} u\big\|_{S^0(I)} \tag{5.22}$$

and interpolation implies that there exist $0 < \theta_1,\ \theta_2 < 1$ such that

$$\|u\|_{X(I)} \lesssim \|u\|^{\theta_1}_{L_{t,x}^{\frac{(d+4)p}{4}}(I\times\mathbb{R}^d)} \big\||\nabla|^{s_c} u\big\|^{1-\theta_1}_{S^0(I)} \tag{5.23}$$

$$\|u\|_{L_{t,x}^{\frac{(d+4)p}{4}}(I\times\mathbb{R}^d)} \lesssim \|u\|^{\theta_2}_{X(I)} \big\||\nabla|^{s_c} u\big\|^{1-\theta_2}_{S^0(I)}, \tag{5.24}$$

Also, as a direct consequence of Hardy-Littlewood-Sobolev inequality, we have the following "exotic Strichartz estimate".

Lemma 5.3 (Exotic Strichartz estimate)**.** *Let I be a compact time interval containing t_0 , then*

$$\Big\|\int_{t_0}^t e^{i(t-s)\Delta^2} F(s)\,ds\Big\|_{X(I)} \lesssim \big\|F\big\|_{Y(I)}. \tag{5.25}$$

Proof. It follows from the dispersive estimate (2.3) that

$$\big\|e^{i(t-s)\Delta^2} F(s)\big\|_{L_x^{r_1}} \lesssim |t-s|^{-\frac{p(d-r_0 s_c)}{4r_0}} \big\|F(s)\big\|_{L_x^{r_1'}}.$$

This together with Hardy-Littlewood-Sobolev inequality yields that

$$\begin{aligned}\Big\|\int_{t_0}^t e^{i(t-s)\Delta^2}F(s)ds\Big\|_{L_t^{q_0}L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim&\Big\|\int_{t_0}^t \|e^{i(t-s)\Delta^2}F(s)\|_{L_x^{r_1}}ds\Big\|_{L_t^{q_0}(I)}\\ \lesssim&\Big\|\int_{t_0}^t |t-s|^{-\frac{p(d-r_0s_c)}{4r_0}}\big\|F(s)\big\|_{L_x^{r_1'}}ds\Big\|_{L_t^{q_0}(I)}\\ \lesssim&\|F\|_{L_t^{\frac{q_0}{p+1}}L_x^{r_1'}(I\times\mathbb{R}^d)}.\end{aligned}$$

□

Lemma 5.4 (Nonlinear estimates). *Let $d\geqslant 9,\ 1\leqslant s_c<2+p$, and I be a time interval. Then*

$$\begin{aligned}&\big\|f_z(u+v)\omega\big\|_{Y(I)}+\big\|f_{\bar z}(u+v)\bar\omega\big\|_{Y(I)}\\ &\lesssim\Big(\|u\|_{X(I)}^{\frac{p(s_c-1)}{s_c}}\big\||\nabla|^{s_c}u\big\|_{S^0(I)}^{\frac{p}{s_c}}+\|v\|_{X(I)}^{\frac{p(s_c-1)}{s_c}}\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{\frac{p}{s_c}}\Big)\|\omega\|_{X(I)},\end{aligned}\tag{5.26}$$

and there exists $\beta\in(0,p)$ such that

$$\begin{aligned}&\big\||\nabla|^{s_c-1}\big[f(u+v)-f(u)\big]\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ \lesssim&\big\||\nabla|^{s_c}u\big\|_{S^0(I)}\Big[\|v\|_{X^0(I)}^p+\|u\|_{X^0(I)}^{p-1+\frac{1}{s_c}}\|v\|_{X^0(I)}^{1-\frac{1}{s_c}}+\big(\|u\|_{X^0(I)}^{p-1+\frac{1}{s_c}}+\|v\|_{X^0(I)}^{p-1+\frac{1}{s_c}}\big)\big\||\nabla|^{s_c}u\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\Big]\\ &+\big\||\nabla|^{s_c}u\big\|_{S^0(I)}\Big(\|v\|_{X^0(I)}^p+\|u\|_{X^0(I)}^{\beta}\|v\|_{X^0(I)}^{p-\beta}\Big)+\big\||\nabla|^{s_c}u\big\|_{S^0(I)}^{\frac{1}{s_c}}\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\|u\|_{X^0(I)}^{1-\frac{1}{s_c}}\|v\|_{X^0(I)}^{p-1+\frac{1}{s_c}}.\end{aligned}\tag{5.27}$$

Proof. **The proof of** (5.26)**:** It suffices to prove the first term on the left-hand side, as the second term can be estimate by the same way. Using (2.10) and (5.22), we derive

$$\begin{aligned}\big\|f_z(u+v)\omega\big\|_{Y(I)}\lesssim&\big\|f_z(u+v)\big\|_{L_t^{\frac{q_0}{p}}L_x^{\frac{r_0d}{p(d-r_0s_c)}}}\|\omega\|_{X(I)}\\ &+\big\||\nabla|^{\frac{p}{2}}f_z(u+v)\big\|_{L_t^{\frac{q_0}{p}}L_x^{\frac{2r_0d}{p(2d-2r_0s_c+r_0)}}}\|\omega\|_{X^0(I)}\\ \lesssim&\Big(\|u+v\|_{X^0(I)}^p+\big\||\nabla|^{\frac{p}{2}}f_z(u+v)\big\|_{L_t^{\frac{q_0}{p}}L_x^{\frac{2r_0d}{p(2d-2r_0s_c+r_0)}}}\Big)\|\omega\|_{X(I)}.\end{aligned}\tag{5.28}$$

Hence, from (5.22), we know that the proof of (5.26) can be reduced to prove

$$\begin{aligned}&\big\||\nabla|^{\frac{p}{2}}f_z(u+v)\big\|_{L_t^{\frac{q_0}{p}}L_x^{\frac{2r_0d}{p(2d-2r_0s_c+r_0)}}}\\ \lesssim&\|u\|_{X(I)}^{\frac{p(s_c-1)}{s_c}}\big\||\nabla|^{s_c}u\big\|_{S^0(I)}^{\frac{p}{s_c}}+\|v\|_{X(I)}^{\frac{p(s_c-1)}{s_c}}\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{\frac{p}{s_c}}.\end{aligned}$$

Case 1: $p\geqslant 1$**.** Using (2.13) and (5.22), we get

$$\big\||\nabla|^{\frac{p}{2}}f_z(u+v)\big\|_{L_t^{\frac{q_0}{p}}L_x^{\frac{2r_0d}{p(2d-2r_0s_c+r_0)}}}\lesssim\|u+v\|_{X^0(I)}^{p-1}\|u+v\|_{X(I)}\lesssim\|u+v\|_{X(I)}^p.$$

Case 2: $p<1$**.** By the fractional chain rule (2.12) with $s=\frac{p}{2}$, $\frac12<\sigma<1$, Hölder inequality, Sobolev embedding and interpolation, we obtain

$$\begin{aligned}\big\||\nabla|^{\frac{p}{2}}f_z(u+v)\big\|_{L_t^{\frac{q_0}{p}}L_x^{\frac{2r_0d}{p(2d-2r_0s_c+r_0)}}}\lesssim&\|u+v\|_{X^0(I)}^{p-\frac{p}{2\sigma}}\big\||\nabla|^{\sigma}(u+v)\big\|^{\frac{p}{2\sigma}}_{L_t^{q_0}L_x^{\frac{r_0d}{d-r_0(s_c-\sigma)}}}\\ \lesssim&\big\||\nabla|^{\sigma}(u+v)\big\|^{p}_{L_t^{q_0}L_x^{\frac{r_0d}{d-r_0(s_c-\sigma)}}}\\ \lesssim&\|u\|_{X(I)}^{\frac{p(s_c-1)}{s_c}}\big\||\nabla|^{s_c}u\big\|_{S^0(I)}^{\frac{p}{s_c}}+\|v\|_{X(I)}^{\frac{p(s_c-1)}{s_c}}\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{\frac{p}{s_c}}.\end{aligned}$$

Plugging this into (5.28), we get (5.26).

The proof of (5.27)**:** When $s_c\in[1,2)$, it is easy to show (5.27) by Lemma 2.3, Hölder's inequality and Sobolev embedding. Now we consider $s_c\in[2,p+2)$. For $p\leqslant 1$. Using the Fundamental Theorem of Calculus and triangle inequality, we deduce that

$$\begin{aligned}&\big\||\nabla|^{s_c-1}\big[f(u+v)-f(u)\big]\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ \lesssim&\big\||\nabla|^{s_c-2}\big[\nabla v\cdot f'(u+v)\big]\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ &+\Big\||\nabla|^{s_c-2}\Big[\nabla u\cdot\big(f'(u+v)-f'(u)\big)\Big]\Big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}.\end{aligned}\tag{5.29}$$

One one hand, by Lemma 2.10, fractional chain rule (2.12), Hölder inequality and interpolation, we obtain

$$\begin{aligned}\big\||\nabla|^{s_c-2}\big[\nabla v\cdot|v|^p\big]\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\lesssim&\big\||\nabla|^{s_c-1}v\big\|_{L_t^{q_2}L_x^{r_2}}\|v\|_{X^0(I)}^p\\ \lesssim&\big\||\nabla|^{s_c}v\big\|_{L_t^{q_2}L_x^{r_3}}\|v\|_{X^0(I)}^p\\ \lesssim&\big\||\nabla|^{s_c}v\big\|_{S^0(I)}\|v\|_{X^0(I)}^p,\end{aligned}$$

where

$$\begin{cases}\frac12=\frac{1}{q_2}+\frac{p}{q_0},\Rightarrow\ q_2=\frac{2q_0}{q_0-p},\\ \frac{d+2}{2}=\frac{d}{r_2}+\frac{(d-p)p}{p+2}\\ \frac{4}{q_2}=d\big(\frac12-\frac{1}{r_3}\big),\Rightarrow\ r_3=\frac{2q_0d}{(d-4)q_0+4p},\\ 1-\frac{d}{r_3}=-\frac{d}{r_2}.\end{cases}$$

Hence

$$\begin{aligned}&\big\||\nabla|^{s_c-2}\big[\nabla v\cdot f'(u+v)\big]\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\ \lesssim&\big\||\nabla|^{s_c}v\big\|_{S^0(I)}\big(\|u\|_{X^0(I)}^p+\|v\|_{X^0(I)}^p\big)+\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{\frac{1}{s_c}}\|v\|_{X^0(I)}^{1-\frac{1}{s_c}}\|u+v\|_{X^0(I)}^{p-1+\frac{1}{s_c}}\big\||\nabla|^{s_c}(u+v)\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\\ \lesssim&\big\||\nabla|^{s_c}v\big\|_{S^0(I)}\Big[\|v\|_{X^0(I)}^p+\|u\|_{X^0(I)}^{p-1+\frac{1}{s_c}}\|v\|_{X^0(I)}^{1-\frac{1}{s_c}}+\big(\|u\|_{X^0(I)}^{p-1+\frac{1}{s_c}}+\|v\|_{X^0(I)}^{p-1+\frac{1}{s_c}}\big)\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\Big].\end{aligned}$$

On the other hand, by Lemma 2.4, Hölder inequality, interpolation and (5.22), one has

$$\begin{aligned}
&\Big\||\nabla|^{s_c-2}\Big[\nabla u\cdot\big(f'(u+v)-f'(u)\big)\Big]\Big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\
\lesssim&\big\||\nabla|^{s_c}u\big\|_{S^0(I)}\|v\|_{X^0(I)}^p+\big\||\nabla|^{s_c}u\big\|_{S^0(I)}\|u\|_{X^0(I)}^{\frac{s_c-1}{\sigma}}\|v\|_{X^0(I)}^{p-\frac{s_c-1}{\sigma}}\\
&+\big\||\nabla|^{s_c}u\big\|_{S^0(I)}^{\frac{1}{s_c}}\|u\|_{X^0(I)}^{1-\frac{1}{s_c}}\big\||\nabla|^{s_c}v\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\|v\|_{X^0(I)}^{p-1+\frac{1}{s_c}},
\end{aligned}$$

where $s_c-1<\sigma p<p$. Letting $\beta:=\frac{s_c-1}{\sigma}$, we derive (5.27) for $p\leqslant 1$.We can iterate the argument presented above to obtain (5.27) for $p>1$. This concludes the proof of this lemma. $\square$

Before we prove the perturbation. We first show the short-time perturbations.

Lemma 5.5 (short-time perturbation)**.** *Let $d\geqslant 9$, p be not an even integer and $1\leqslant s_c<2+p$. Let I be a compact time interval and $u,\ \tilde u$ satisfy*

$$\begin{aligned}
(i\partial_t+\Delta^2)u=&-f(u)+eq(u)\\
(i\partial_t+\Delta^2)\tilde u=&-f(\tilde u)+eq(\tilde u)
\end{aligned}$$

for some function $eq(u),eq(\tilde u)$, and $f(u)=|u|^pu$. Assume that for some constants $E>0$, we have

$$\|u\|_{L_t^\infty(I;\dot H_x^{s_c}(\mathbb{R}^d))}+\|\tilde u\|_{L_t^\infty(I;\dot H_x^{s_c}(\mathbb{R}^d))}\leqslant E. \tag{5.30}$$

Moreover, for $t_0\in I$, and assume that smallness conditions

$$\|\tilde u\|_{X(I)}\leqslant\delta \tag{5.31}$$

$$\big\|u(t_0)-\tilde u(t_0)\big\|_{\dot H^{s_c}}\leqslant\varepsilon \tag{5.32}$$

$$\big\||\nabla|^{s_c-1}\big(eq(u),eq(\tilde u)\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\leqslant\varepsilon \tag{5.33}$$

for some small $0<\varepsilon<\varepsilon_1=\varepsilon_1(M,E)$ and $0<\varepsilon<\varepsilon_0(E)$. Then we conclude that

$$\|u-\tilde u\|_{X(I)}\lesssim\varepsilon \tag{5.34}$$

$$\big\||\nabla|^{s_c}(u-\tilde u)\big\|_{S^0(I)}\lesssim\varepsilon^{c(d,p)} \tag{5.35}$$

$$\big\||\nabla|^{s_c}u\big\|_{S^0(I)}\lesssim E \tag{5.36}$$

$$\big\|f(u)-f(\tilde u)\big\|_{Y(I)}\lesssim\varepsilon \tag{5.37}$$

$$\big\||\nabla|^{s_c-1}\big(f(u)-f(\tilde u)\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\lesssim\varepsilon^{c(d,p)}, \tag{5.38}$$

for some positive constant $c(d,p)$.

Proof. **Step 1: We claim that $\big\||\nabla|^{s_c}\tilde u\big\|_{S^0(I)}\lesssim E$.**

Indeed, using Strichartz estimate, Corollary 2.1, (5.31) and (5.33), we get

$$\begin{aligned}
\big\||\nabla|^{s_c}\tilde u\big\|_{S^0(I)}\lesssim&\|\tilde u\|_{L_t^\infty\dot H^{s_c}}+\big\||\nabla|^{s_c}f(\tilde u)\big\|_{N^0(I)}+\big\||\nabla|^{s_c}e\big\|_{N^0(I)}\\
\lesssim&E+\|u\|^p_{L_{t,x}^{\frac{(d+4)p}{4}}(I\times\mathbb{R}^d)}\big\||\nabla|^{s_c}\tilde u\big\|_{S^0(I)}+\varepsilon\\
\lesssim&E+\delta^{p\theta_2}\big\||\nabla|^{s_c}\tilde u\big\|_{S^0(I)}^{1+p(1-\theta_2)}+\varepsilon.
\end{aligned}$$

Hence, by the standard bootstrap argument, and choosing $\delta,\ \varepsilon_0$ to be sufficiently small, we obtain

$$\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I)} \lesssim E. \tag{5.39}$$

Step 2: We claim that $\|u\|_{X(I)} \lesssim \delta$.

By Lemma 5.3, (5.21), (5.31) and (5.33), one has

$$\big\|e^{i(t-t_0)\Delta^2}\tilde{u}(t_0)\big\|_{X(I)} \lesssim \|\tilde{u}\|_{X(I)} + \|f(\tilde{u})\|_{Y(I)} + \big\||\nabla|^{s_c} eq(\tilde{u})\big\|_{N^0(I)} \lesssim \delta + \delta^{p+1} + \varepsilon \lesssim \delta.$$

Combining this with the triangle inequality, (5.22), Strichartz estimate and (5.32), we derive

$$\begin{aligned}\big\|e^{i(t-t_0)\Delta^2}u(t_0)\big\|_{X(I)} \lesssim& \big\|e^{i(t-t_0)\Delta^2}\tilde{u}(t_0)\big\|_{X(I)} + \big\|e^{i(t-t_0)\Delta^2}(u-\tilde{u})(t_0)\big\|_{X(I)}\\ \lesssim& \delta + \big\|u(t_0)-\tilde{u}(t_0)\big\|_{\dot{H}^{s_c}} \lesssim \delta.\end{aligned}$$

On the other hand, by Lemma 5.3 and Lemma 5.4, we get

$$\|u\|_{X(I)} \lesssim \big\|e^{i(t-t_0)\Delta^2}u(t_0)\big\|_{X(I)} + \|f(u)\|_{Y(I)} + \big\||\nabla|^{s_c-1}eq(u)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} \lesssim \delta + \|u\|_{X(I)}^{p+1}.$$

Thus, we obtain by the bootstrap argument

$$\|u\|_{X(I)} \lesssim \delta. \tag{5.40}$$

Step 3: Next we prove the following iteration formula

$$\|\omega\|_{X(I)} \lesssim \varepsilon + \big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{\frac{p}{s_c}} \|\omega\|_{X(I)}^{1+\frac{p(s_c-1)}{s_c}}, \tag{5.41}$$

$$\big\||\nabla|^{s_c}\omega\big\|_{S^0(I)} \lesssim \varepsilon + \|\omega\|_{X(I)}^{p-\beta} + \big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{1-\frac{1}{s_c}} \|\omega\|_{X(I)}^{p-1+\frac{1}{s_c}}, \tag{5.42}$$

where $\omega = u - \tilde{u}$ satisfies the difference equation

$$\begin{cases} i\omega_t + \Delta^2\omega = -f(\tilde{u}+\omega) + f(\tilde{u}) + eq(u) - eq(\tilde{u}),\\ \omega(t_0,x) = u(t_0,x) - \tilde{u}(t_0,x) \in \dot{H}^{s_c}(\mathbb{R}^d).\end{cases} \tag{5.43}$$

Using Lemma 5.3, Strichartz estimate, (5.31) and (5.32), we get

$$\begin{aligned}\|\omega\|_{X(I)} \lesssim& \big\|u(t_0)-\tilde{u}(t_0)\big\|_{\dot{H}^{s_c}} + \big\||\nabla|^{s_c-1}\big(eq(u),eq(\tilde{u})\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)} + \big\|f(u)-f(\tilde{u})\big\|_{Y(I)}\\ \lesssim& \varepsilon + \big\|f(u)-f(\tilde{u})\big\|_{Y(I)}.\end{aligned} \tag{5.44}$$

The estimate of $\big\|f(u)-f(\tilde{u})\big\|_{Y(I)}$: From Lemma 5.4, (5.31) and Step 1: (5.39), we know that

$$\begin{aligned}\big\|f(u)-f(\tilde{u})\big\|_{Y(I)} \lesssim& \Big[\|\tilde{u}\|_{X(I)}^{\frac{p(s_c-1)}{s_c}} \big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I)}^{\frac{p}{s_c}} + \|\omega\|_{X(I)}^{\frac{p(s_c-1)}{s_c}} \big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{\frac{p}{s_c}}\Big] \|\omega\|_{X(I)}\\ \lesssim& \delta^{\frac{p(s_c-1)}{s_c}} E^{\frac{p}{s_c}} \|\omega\|_{X(I)} + \big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{\frac{p}{s_c}} \|\omega\|_{X(I)}^{1+\frac{p(s_c-1)}{s_c}}.\end{aligned} \tag{5.45}$$

Plugging this into (5.44), and taking δ sufficiently small, we have

$$\|\omega\|_{X(I)} \lesssim \varepsilon + \big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{\frac{p}{s_c}} \|\omega\|_{X(I)}^{1+\frac{p(s_c-1)}{s_c}}. \tag{5.46}$$

This is (5.41).

On the other hand, using Strichartz estimate, (5.32) and (5.33), we obtain

$$\begin{aligned}
&\big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}\\
\lesssim&\big\|u(t_0)-\tilde{u}(t_0)\big\|_{\dot{H}^{s_c}}+\big\||\nabla|^{s_c-1}\big(eq(u),eq(\tilde{u})\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\
&+\big\||\nabla|^{s_c-1}\big(f(u)-f(\tilde{u})\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\
\lesssim&\varepsilon+\big\||\nabla|^{s_c-1}\big(f(u)-f(\tilde{u})\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}.
\end{aligned}\tag{5.47}$$

The estimate of $\big\||\nabla|^{s_c-1}\big(f(u)-f(\tilde{u})\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}$: By (5.27), (5.31), Step 1: (5.39) and Step 2: (5.40), one has

$$\begin{aligned}
&\big\||\nabla|^{s_c-1}\big(f(u)-f(\tilde{u})\big)\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I\times\mathbb{R}^d)}\\
\lesssim&\big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}\big(\delta+\delta^{p-1+\frac{1}{s_c}}E^{1-\frac{1}{s_c}}\big)+E\delta^{\beta}\|\omega\|_{X(I)}^{p-\beta}+\delta^{1-\frac{1}{s_c}}E^{\frac{1}{s_c}}\big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\|\omega\|_{X(I)}^{p-1+\frac{1}{s_c}}.
\end{aligned}\tag{5.48}$$

Plugging this into (5.47), we have

$$\big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}\lesssim\varepsilon+\|\omega\|_{X(I)}^{p-\beta}+\big\||\nabla|^{s_c}\omega\big\|_{S^0(I)}^{1-\frac{1}{s_c}}\|\omega\|_{X(I)}^{p-1+\frac{1}{s_c}}.\tag{5.49}$$

This is (5.42). Putting this into (5.46), and by the bootstrap argument, we conclude (5.34) and (5.35).

And then, it is easy to show (5.36) by the triangle inequality, (5.39) and (5.35).

Using (5.34), (5.35), (5.45) and (5.45), we obtain (5.37) and (5.38). We concludes the proof of this lemma. □

Now we turn to prove Lemma 5.2.

The proof of Lemma 5.2: First, we claim

$$\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I)}\leqslant C(E,M).\tag{5.50}$$

In fact, from the hypothesis (5.15), we know that one can subdivide time interval I by $I=\cup_j I_j,\ I_j=[t_j,t_{j+1}],\ 0\leqslant j<J_0=J_0(M,\eta)$, such that

$$\|\tilde{u}\|_{L_{t,x}^{\frac{p(d+4)}{4}}(I_j\times\mathbb{R}^d)}\leqslant\eta,$$

where $\eta>0$ is sufficiently small to be determined. Using Strichartz estimate, fractional chain rule (2.1), (5.15) and (5.17), we get

$$\begin{aligned}
\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I_j)}\lesssim&\|\tilde{u}(t_j)\|_{\dot{H}^{s_c}}+\big\||\nabla|^{s_c-1}eq(\tilde{u})\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I_j\times\mathbb{R}^d)}+\big\||\nabla|^{s_c-1}f(\tilde{u})\big\|_{L_t^2L_x^{\frac{2d}{d+2}}(I_j\times\mathbb{R}^d)}\\
\lesssim&E+\varepsilon+\|\tilde{u}\|^p_{L_{t,x}^{\frac{p(d+4)}{4}}(I_j\times\mathbb{R}^d)}\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I_j)}\\
\lesssim&E+\varepsilon+\eta^p\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I_j)}.
\end{aligned}$$

Thus, by the bootstrap argument, we have

$$\big\||\nabla|^{s_c}\tilde{u}\big\|_{S^0(I_j)}\lesssim E+\varepsilon.$$

Summing the above bound over all subinteraval I_j, we get the claim (5.50). In particular, from Sobolev embedding (5.22), we know that

$$\|\tilde{u}\|_{X(I)} \leqslant C(E, M). \tag{5.51}$$

Hence, we can subdivide time interval I by $I = \cup_j I_j,\ I_j = [t_j, t_{j+1}],\ 0 \leqslant j < J_1 = J_1(M, \eta)$, such that

$$\|\tilde{u}\|_{X(I_j)} \leqslant \delta,$$

where δ is as in Lemma 5.5.

Therefore, we can apply Lemma 5.5 to each I_j . And so, $\forall\ 0 \leqslant j < J_1,\ 0 < \varepsilon < \varepsilon_1$,

$$\begin{aligned}
\|u - \tilde{u}\|_{X(I_j)} &\lesssim \varepsilon \\
\big\||\nabla|^{s_c}(u - \tilde{u})\big\|_{S^0(I_j)} &\lesssim \varepsilon^{c(d,p)} \\
\big\||\nabla|^{s_c} u\big\|_{S^0(I_j)} &\lesssim E \\
\big\|f(u) - f(\tilde{u})\big\|_{Y(I_j)} &\lesssim \varepsilon \\
\big\||\nabla|^{s_c-1}\big(f(u) - f(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}(I_j \times \mathbb{R}^d)} &\lesssim \varepsilon^{c(d,p)},
\end{aligned} \tag{5.52}$$

provided that one can prove for any $0 \leqslant j < J_1$

$$\|u(t_j) - \tilde{u}(t_j)\|_{\dot{H}_x^{s_c}(I_j)} \leqslant C_j \varepsilon^{c(d,p)^j} \leqslant \varepsilon_0. \tag{5.53}$$

Indeed, by Strichartz estimate and the inductive hypothesis, one has

$$\begin{aligned}
&\|u(t_j) - \tilde{u}(t_j)\|_{\dot{H}_x^{s_c}(I_j)} \\
\lesssim &\|u_0 - \tilde{u}_0\|_{\dot{H}_x^{s_c}(I_j)} + \big\||\nabla|^{s_c-1}\big(eq(u), eq(\tilde{u})\big)\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,t_j] \times \mathbb{R}^d)} \\
&+ \big\||\nabla|^{s_c-1}\big[f(\tilde{u} + \omega) - f(\tilde{u})\big]\big\|_{L_t^2 L_x^{\frac{2d}{d+2}}([0,t_j] \times \mathbb{R}^d)} \\
\lesssim &\varepsilon + \sum_{k=0}^{j-1} C_k \varepsilon^{c(d,p)^k}.
\end{aligned}$$

Taking ε_1 sufficiently small compared to ε_0, we derive (5.53).

Summing the bounds in (5.52) over all subintervals I_j, we conclude Lemma 5.2.

Acknowledgements The authors were supported by the NSF of China under grant No.11171033, 11231006.

References

[1] M. Ben-Artzi, H. Koch and J.C. Saut. Dispersion estimates for fourth order Schrödinger equations, C.R.A.S., 330, Serie 1, (2000) 87-92.

[2] J. Bourgain, Global well-posedness of defocusing 3D critical NLS in the radial case. J. Amer. Math. Soc., 12(1999), 145-171.

[3] M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal., 100 (1991), 87-109. MR1124294.

[4] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and scattering for the energy-cirtical nonlinear Schrödinger equation in $\mathbb{R}^3$. Annals of Math., 167 (2008), 767-865.

[5] B. Dodson, Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d \geqslant 3$. J. Amer. Math. Soc., 25 (2012), 429-463. MR2869023.

[6] B. Dodson, Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$. Preprint arXiv:1006.1375.
[7] B. Dodson, Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 1$. Preprint arXiv:1010.0040.
[8] B. Dodson, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state. Preprint arXiv:1104.1114.
[9] G. Fibich, B. Ilan and G. Papanicolaou, Self-focusing with fourth order dispersion. SIAM J. Appl. Math., 62, No 4, (2002) 1437-1462.
[10] J. Ginibre and G. Velo, Scattering theory in the energy space for a class of Hartree equations, Nonlinear wave equations (Providence, RI, 1998), 29-60, Contemp. Math., 263, Amer. Math. Soc., Providence, RI, 2000.
[11] B. Guo and B. Wang, The global Cauchy problem and scattering of solutions for nonlinear Schrödinger equations in H^s, Diff. Int. Equ. 15 No 9 (2002) 1073-1083.
[12] V.I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrodinger-type equations. Phys. Rev. E53,2 (1996) 1336- 1339.
[13] V.I. Karpman and A.G. Shagalov, Stability of soliton described by nonlinear Schrodinger-type equations with higher-order dispersion. Phys. Rev. D. 144 (2000) 194-210.
[14] M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math., 120:5 (1998), 955-980.
[15] C. Kenig and F. Merle, Global well-posedness, scattering, and blow-up for the energy-critical focusing nonlinear Schrödinger equation in the radial case, Invent. Math., 166:3 (2006), 645-675.
[16] C. Kenig and F. Merle, Scattering for $\dot{H}^{\frac{1}{2}}$ bounded solutions to the cubic, defocusing NLS in 3 dimensions. Trans. Amer. Math. Soc., 362 (2010), 1937-1962. MR2574882.
[17] S. Keraani, On the blow-up phenomenon of the critical nonlinear Schrödinger equation. J. Funct. Anal., 235 (2006), 171-192.
[18] R. Killip, T. Tao, and M. Visan, The cubic nonlinear Schrödinger equation in two dimensions with radial data. J. Eur. Math. Soc., 11 (2009), 1203-1258. MR2557134.
[19] R. Killip and M. Visan, Energy-supercritical NLS: critical $\dot{H}^s$-bounds imply scattering. Comm. Partial Differential Equations 35 (2010), 945-987. MR2753625.
[20] R. Killip and M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. Amer. J. Math., 132 (2010), 361-424. MR2654778.
[21] R. Killip and M. Visan, The radial defocusing energy-supercritical nonlinear wave equation in all space dimensions. Proc. Amer. Math. Soc., 139 (2011), 1805-1817.
[22] R. Killip and M. Visan, The defocusing energy-supercritical nonlinear wave equation in three space dimensions, Trans. Amer. Math. Soc., 363 (2011), 3893-3934.
[23] R. Killip and M. Visan, Nonlinear Schrödinger equations at critical regularity. Lecture notes prepared for Clay Mathematics Institute Summer School, Zürich, Switzerland, 2008.
[24] R. Killip, M. Visan, and X. Zhang. The mass-critical nonlinear Schrödinger equation with radial data in dimensions three and higher. Analysis and Partial Differential Equations, 1, no. 2 (2008) 229-266. MR2472890.
[25] C. Miao, G. Xu, and L. Zhao. Global well-posedness and scattering for the focusing energy critical nonlinear Schrödinger equations of fourth order in the radial case, J. Differential Equations, 246 (2009) 3715-3749.
[26] C. Miao, G. Xu, and L. Zhao. Global well-posedness and scattering for the defocusing energy critical nonlinear Schrödinger equations of fourth order in dimensions $d \geqslant 9$, J. Differential Equations, 251 (2011) 3381-3402.
[27] C. Miao, and B. Zhang. Global well-posedness of the Cauchy problem for nonlinear Schrödinger-type equations, Discrete Contin. Dyn. Syst., 17 (2007) 181-200.
[28] C. Miao, H. Wu and J. Zhang, Scattering theory below energy for the cubic fourth-order Schrödinger equation, arXiv:0907.5047v2.
[29] C. Miao and B. Zhang, Global well-posedness of the Cauchy problem for nonlinear Schrödinger-type equations, Discrete Contin. Dyn. Syst. 17 (2007) 181-200.
[30] J. Murphy, Inter-critical NLS: critical $\dot{H}^s_x$-bounds imply scattering. arXiv: 1209.4582V1.
[31] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case. Dynamics of PDE, Vol.4, No.3, 197-225, 2007.

[32] B. Pausader, The cubic fourth-order Schrödinger equation, J. Funct. Anal., 256 (2009) 2473–2517.
[33] B. Pausader, The focusing energy-critical fourth-order Schrödinger equation with radial data, DCDS-A, 24 (2009), 4, 1275–1292.
[34] B. Pausader and S. Shao, The mass-critical fourth-order Schrödinger equation in high dimensions, J. Hyp. Diff. Equ., 7 (2010) 65-705.
[35] J. Segata, Modified wave operators for the fourth-order non-linear Schrödingertype equation with cubic non-linearity. Math. Meth. in the Appl. Sci. 26 (2006) 1785-1800.
[36] T. Tao and M. Visan, Stability of energy-critical nonlinear Schrödinger equations in high dimensions, Electron. J. Diff. Eqns., 118 (2005) 1-28.
[37] T. Tao, M. Visan, and X. Zhang, Global well-posedness and scattering for the mass-critical nonlinear Schrödinger equation for radial data in high dimensions. Duke Math. J., 140 (2007) 165-202. MR2355070.
[38] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. Ph. D Thesis, UCLA, 2006.
[39] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. Duke Math. J., 138 (2007) 281-374. MR2318286.
[40] M. Visan, Global well-posedness and scattering for the defocusing cubic NLS in four dimensions. Int. Math. Res. Not. 2011 (2011), doi: 10.1093/imrn/rnr051.
[41] S. Xia and B. Pausader, Scattering theory for the fourth-order Schrödinger equation in low dimension. To appear in Nonlinearity.
[42] J. Zhang and J. Zheng, Multilinear Estimates and Well-posedness for the Vortex Filament Fourth-order Schrödinger Equations. To appaer in Mathmatical methods in applied science.
[43] J. Zheng, Well-posedness for the Fourth-order Schrödinger Equations with Quadratic Nonlinearity, Advances in Differential Equations, V16, no 5-6 (2011), 467-486.

Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing, China, 100088;
E-mail address: miao_changxing@iapcm.ac.cn

The Graduate School of China Academy of Engineering Physics, P. O. Box 2101, Beijing, China, 100088
E-mail address: zhengjiqiang@gmail.com